TABLE OF CONTENTS for "BIG BABY"

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F	Parametric and Polar Forms in Area, Volume, Arc Length, and Surface Area (including Vectors)
H	Sequences and Series

The AP Calculus Exam

Be sure to pace yourself during each section.

Section I-A – non-calculator multiple choice – 28 questions – 55 minutes – about 2 minutes per problem
Section I-B – calculator multiple choice – 17 question – 50 minutes – about 3 minutes per problem
Section II-A – calculator short answer – 2 questions – 30 minutes – 15 minutes per problem
Section II-B – non-calculator short answer – 4 questions – 60 minutes – 15 minutes per problem – you will also have the first two short answer questions during this time period

Calculator Uses on Previous AP <u>Short Answer</u> Problems

Time/Pacing – 15 minutes per problem – all short answer questions are graded out of 9 points

1969 – 1992	no calculator allowed
1993 – 1994	all 6 short answer questions allowed a scientific calculator
1995 – 1999	all 6 short answer questions allowed a graphing calculator
2000 – 2010	the first 3 short answer questions allowed a graphing calculator, the last 3 questions were non-calculator
2011 – now	the first 2 short answer questions allowed a graphing calculator, the last 4 questions were non-calculator

SECTION A - "The Building Blocks for the Barn" (a.k.a. Review of Functions)

Overview

In calculus, functions are the building blocks of all ideas. Functions are special relationships between two quantities where one quantity depends on the other quantity. In previous mathematics courses, you probably dealt with formulas, equations, etc., finding values for a table, graphing these values, and possibly writing equations that modeled real-world situations. Earlier courses in mathematics gave you the skills needed in working with variables that are the foundation for analysis of functions, which is what calculus is about.

There are some terms and definitions that you should know that will be included in all discussions about functions. Those terms are:

Relation – a set of ordered pairs
Domain – the set of all values for which the function is defined
Range – the set of all values that a function takes on using all values in the domain
Function – a relation such that each member of the domain has one and only one member of the range

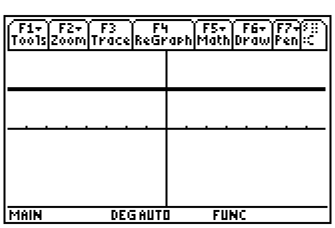
Often, functions are written using *x* and *y* from the Cartesian coordinate system. That is what your graphing calculator uses for graphing functions. Don't think that every equation or formula is a function; they are not. However, functions have special qualities and require special analysis. Usually the *x*-coordinate is called the independent variable, and the *y*-coordinate is called the dependent variable. This is due to the fact that *y* "depends" on *x*. Another way of saying this is that *y* varies, changes, etc. as *x* varies, changes, etc. It is very important to remember which variable is the independent variable and which variable is the dependent variable for a particular problem.

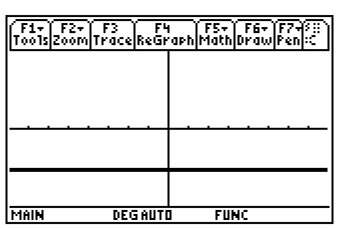
There are many ways to represent functions such as with words, tables, graphs, and, of course, formulas.   This fits right in with the four ways that you will work: verbally (written descriptions since this is an on-line course), numerically, graphically, and analytically. Many of your activities will highlight several of these methods.

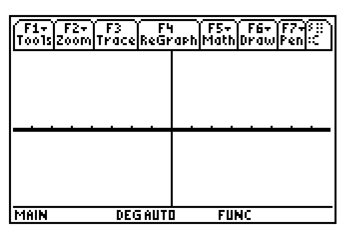
We will review a type of function and discuss important characteristics of these functions. For each type of function, you will need to know the domain, range, generic graph, and properties of each. Be sure that you look for comparisons and contrasts with each type of function. Now, let's get started with the easiest function there is, the constant function.

Constant Function

The constant function $f(x) = c$ has its domain as all real numbers and its range consists of the single number *c*. The graph of a constant function is a horizontal line. Possible graphs of constant functions lie below. In the first graph, the constant is a positive number. In the second graph the constant is a negative number. In the third graph the constant is zero; this graph is a graph of the *x*-axis.



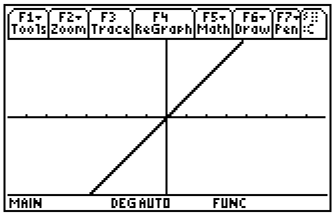


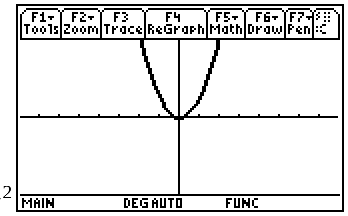


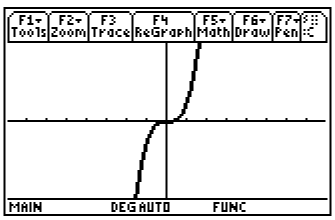
Power Function

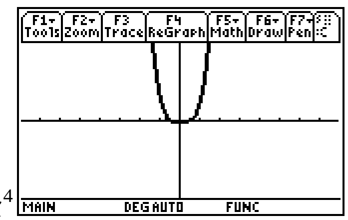
The power function $f(x) = x^a$, where a is a constant has several cases to consider.

(1) If a is a positive integer (like 1, 2, 3, …), then the domain is all real numbers. If a is even, then the range will be numbers greater than or equal to zero. If a is odd, then the range will be all real numbers. You have probably graphed many examples of these functions. Their graphs are lines, parabolas, s-shaped curves, etc.

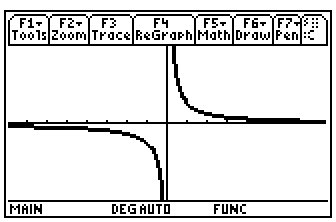
$y = x$ 

$y = x^2$ 

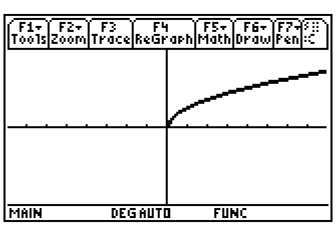
$y = x^3$ 

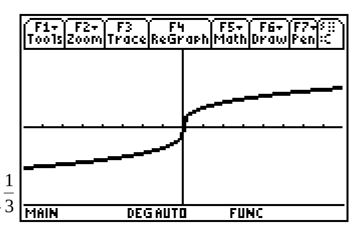
$y = x^4$ 

(2) If a is –1, then the power function is $f(x) = \dfrac{1}{x}$, which is an equilateral hyperbola whose asymptotes are the x- and y-axes. The domain and range for this function are all real numbers except 0.

$y = \dfrac{1}{x}$ 

(3) If a is of the form $\dfrac{1}{b}$, where b is a positive integer, then this function is called a **root** function. For example, $f(x) = x^{\frac{1}{2}}$ is the same function as $f(x) = \sqrt{x}$. When b is even, then the domain and the range of the function are all numbers greater than or equal to zero. When b is odd, then the domain and range of the function are all real numbers.

$y = \sqrt{x} = x^{\frac{1}{2}}$ 

$y = \sqrt[3]{x} = x^{\frac{1}{3}}$ 

Polynomial Functions

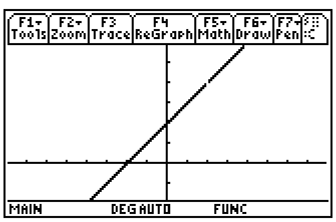
A polynomial function is $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$, where n is a **nonnegative integer** and the numbers $a_0, a_1, a_2, ..., a_n$ are constants called the coefficients of the polynomial. The domain of this function is all real numbers. If the leading coefficient $a_n \neq 0$, then the degree of the polynomial is n. There are several special terms to describe the degree of polynomials. They are:

Linear – degree is 1 – graph is a line – has the form of $f(x) = mx + b$, where m is the slope and b is the y-intercept.
Quadratic – degree is 2 – graph is a parabola – has the form of $f(x) = ax^2 + bx + c$
Cubic – degree is 3 – graph is a s-shaped curve – has the form of $f(x) = ax^3 + bx^2 + cx + d$
Quartic – degree is 4
Quintic – degree is 5, and so on.

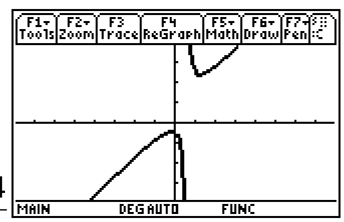
Rational Functions

A rational function is the ratio of two polynomials and has the form of $f(x) = \frac{u(x)}{v(x)}$, where *u(x)* and *v(x)* are polynomials. The domain of this function consists of all values such that $v(x) \neq 0$. Strange things can happen to rational functions. Look at the following examples and try to discover what is occurring. Graph the functions on your calculator and change the viewing window by zooming in and zooming out. **Remember, zooming in and zooming out changes the scale on the *x*- and *y*-axes.**

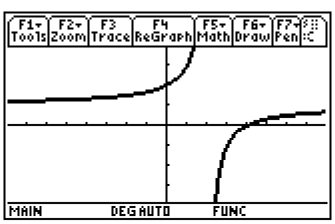
$y = \frac{x^2 - 4}{x - 2}$



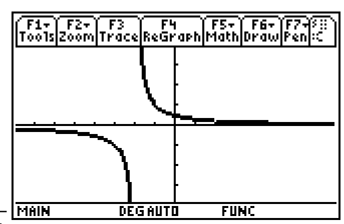
$y = \frac{x^2 + 4}{x - 2}$



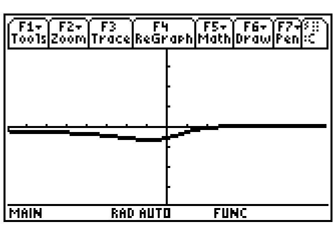
$y = \frac{x - 4}{x - 2}$



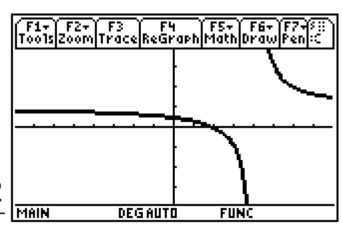
$y = \frac{x - 2}{x^2 - 4}$



$y = \frac{x - 2}{x^2 + 4}$



$y = \frac{x - 2}{x - 4}$



Algebraic Functions

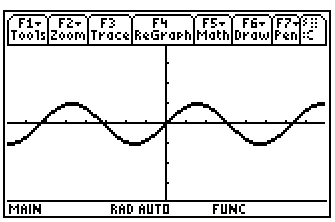
An algebraic function is a function constructed by using algebraic operations (addition, subtraction, multiplication, division, roots, powers, etc) starting with polynomials. All rational functions are automatically algebraic functions.

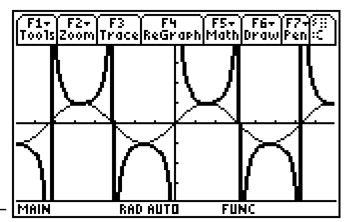
Transcendental Functions

These are functions that are not algebraic. The group of transcendental functions include trigonometric, inverse trigonometric, exponential, and logarithmic functions, but it also include infinitely more functions that have no specific names.

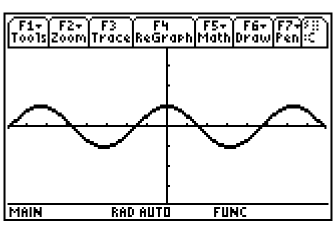
Trigonometric Functions

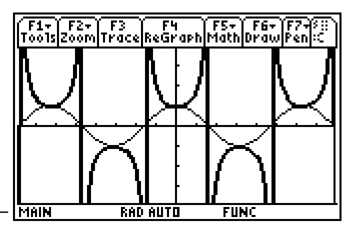
Trigonometric functions include all the six trigonometric ratios defined from the right triangle such as sine, cosine, tangent, cotangent, secant, and cosecant. In calculus, you should always use radian measure since radians are real numbers and degrees are not real numbers. You should know the six trigonometric values for the principal angles in the first quadrant and how they related to the other quadrants. These angles include: $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$, and $\frac{\pi}{2}$. You should also know the domains and ranges for all six trigonometric functions. These functions are periodic, which means that they repeat in a predictable pattern, or period. The period of sine, cosine, secant, and cosecant is 2π; the period of tangent and cotangent is π. You should also know the reciprocal and opposite angle relationships of these functions. These things will be reviewed in an activity in this unit. This activity is not an assignment; however, it will help you to print this activity and keep for a reference. Also, the media center will contain a table of trigonometric identities which you might want to print as a reference.

$y = \sin x$


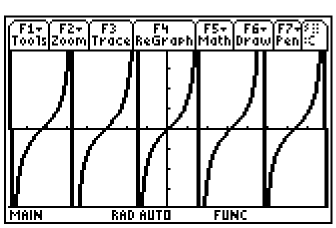
$y = \csc x = \frac{1}{\sin x}$


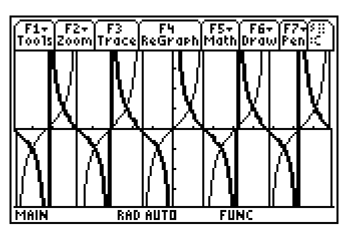
Notice that the reciprocal graph is in bold and the original is light.

$y = \cos x$


$y = \sec x = \frac{1}{\cos x}$


Notice that the reciprocal graph is in bold and the original is light.

$y = \tan x = \frac{\sin x}{\cos x}$


$y = \cot x = \frac{1}{\tan x}$


Trigonometric Graph Transformations

General Form of a trig equation (in sine or cosine) is: $y = a\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}\left(b\left(x-c\right)\right)+d$

a - affects the <u>amplitude</u> (distance above and below the middle line) – steeper and deeper

b– affects the <u>period</u> – *b* is the number of complete cycles in 2π units

If period was 2π, then new period = $\frac{2\pi}{b}$

c – affects <u>horizontal</u> shift

If $c<0$, move <u>right</u> *c* units (think *x* – *c*)

If $c>0$, move <u>left</u> *c* units (think *x* + *c*)

d – affects vertical shift

If $d>0$, move <u>up</u> *d* units

If $d<0$, move <u>down</u> *d* units

Highest point is *d* + *a*; lowest point is *d* – *a*.

When graphing, remember *y* = *d* is the middle line.

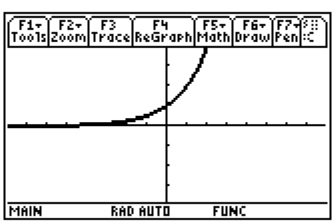
***Exception for tangent** and **cotangent**

b– affects the <u>period</u> – *b* is the number of complete cycles in π units

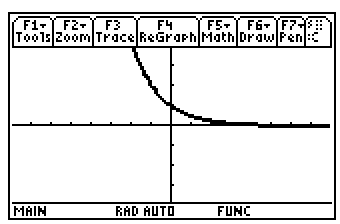
Exponential Functions

An exponential function has the form $f(x)=b^x$, where *b* is a **positive** constant. The domain of this function is all real numbers, and the range is all positive real numbers. Some examples of exponential functions are graphed below. There is a special exponential function, called the natural exponential function, whose base is *e*, a number similar to π, but less than 3, and whose decimal representation is non-terminating and non-repeating. A famous mathematics named Euler had the privilege of giving the first letter of his last name to this very important number.

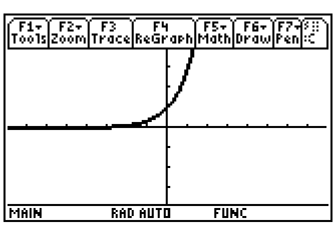
$y = 2^x$



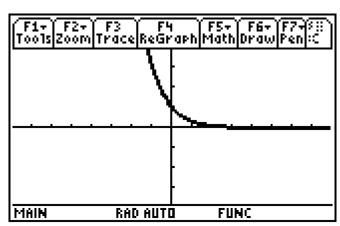
$y = \left(\frac{1}{2}\right)^x = 2^{-x}$



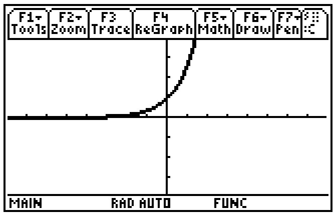
$y = 3^x$



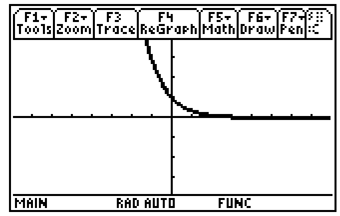
$y = \left(\frac{1}{3}\right)^x = 3^{-x}$



$y = e^x$



$y = e^{-x}$

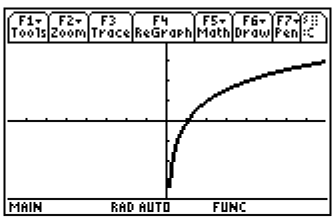


Investigate other exponential functions using your graphing calculator. Look for similarities and differences when you negate the power. Also, be ready to investigate the history and mathematical contributions of Euler.

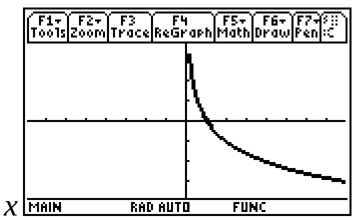
Logarithmic Functions

A logarithmic function has the form $f(x) = \log_b x$, where *b* is a positive constant. This function is the inverse of the exponential function, $f(x) = b^x$. An easy conversion is: $n = b^x \Leftrightarrow \log_b n = x$. The domain of this function is all positive real numbers, and the range is all real numbers. Notice that the logarithm of a number is nothing more than an exponent. That is why the rules for exponents are very similar to the rules for logarithms. The following graphs are of logarithmic functions with varying bases. A special logarithmic function, the natural logarithmic function, has as its base the number *e*. One reason why the natural exponential and natural logarithmic functions are so important is their use in many practical applications such as compound interest, growth and decay, heating and cooling, force of earthquakes, and so on.

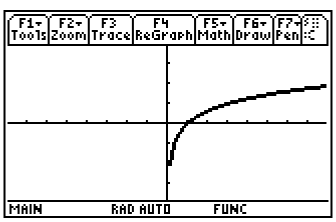
$y = \log_2 x$



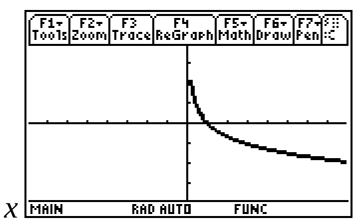
$y = \log_{\frac{1}{2}} x$



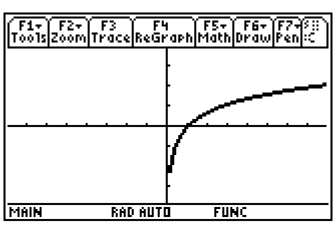
$y = \log_3 x$



$y = \log_{\frac{1}{3}} x$



$y = \log_e x = \ln x$



Notice how the logarithmic functions get closer and closer to the x-axis but never touch it. This is because there is no answer to the following expression: $\log_b 0 = ?$ This logarithmic expression can be converted to: $b^? = 0$, and clearly this expression has no solution. Also, logarithms are defined for only **positive** bases.

The following chart is about two special logarithmic and exponential functions.

	$\ln x$	e^x
Domain	$(0, \infty)$	All reals
Range	All reals	$(0, \infty)$
Concavity	Concave down	Concave up
Increasing or decreasing	Always increasing	Always increasing
Continuous	$(0, \infty)$ - not defined for $x \le 0$	All reals
Asymptotes	$x = 0$	$y = 0$

Properties of Logarithms

$\log_b n = x \leftrightarrow b^x = n, \quad b > 0, \quad n > 0$

$\log_b xy = \log_b x + \log_b y$

$\log_b \frac{x}{y} = \log_b x - \log_b y$

$\log_b x^y = y \log_b x$

$\log_b b = 1$

$\log_b 1 = 0$

$b^{\log_b n} = n$

$\log_b n = \frac{\log_a n}{\log_a b}$

$\log_b n = \frac{1}{\log_n b}$

$\log_{b^a} n = \frac{1}{a} \log_b n$

Graphs

Graphs are a great way of communicating information about functions. There are some things to look for in interpreting graphs. Does the value of the dependent variable increase or decrease when the value of the independent variable increases? What does this mean in the context you are thinking about? Does this make sense? What is the value of the dependent variable when the value of the independent variable is zero? Does this make sense? Does the value of the dependent variable change at a steady rate? If not, how does it change? Is the change faster at first and slower later on, or is it slow at first and faster later?

You will be drawing graphs of functions using many different skills. One of which is the graphing calculator. Remember, for the graphing calculator, you must write your equation in $y = mx + b$ form.

A viewing window is a rectangular portion of the graph of a function. A valuable viewing window needs to show the "essential details" of a graph. Some viewing windows are not square; therefore, the graph is "skewed." Be careful to know what viewing window you are using. Some windows can be misleading!

Symmetry

There are four types of symmetry that you will need to know. They are: symmetry in the *x*-axis, symmetry in the *y*-axis, symmetry in the line $y = x$, and symmetry in the origin.

Symmetry in the *x*-axis means that *(x, -y)* is on the graph whenever *(x, y)* is.
Symmetry in the *y*-axis means that *(-x, y)* is on the graph whenever *(x, y)* is.
Symmetry in the line $y = x$ means that *(y, x)* is on the graph whenever *(x, y)* is.
Symmetry in the origin means that *(-x, -y)* is on the graph whenever *(x, y)* is.

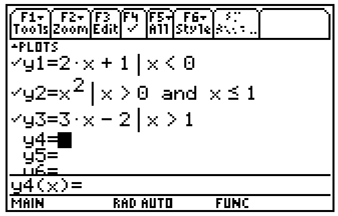
Odd and Even Functions

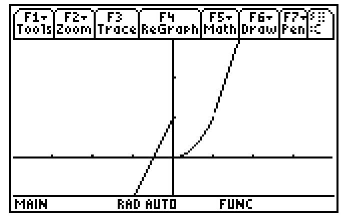
The function *f(x)* is an odd function if *f(-x) = - f(x)*. Odd functions are symmetric in the origin.
The function *f(x)* is an even function if *f(-x) = f(x)*. Even functions are symmetric in the *y*-axis.
There are also functions that are neither odd nor even.

Piece-wise Functions

A piece-wise function is composed by applying different formulas to different parts of a restricted domain for the function. For example, the following function is a piece-wise function:

$$f(x)=\begin{cases} 2x+1 & x<0 \\ 1 & x=0 \\ x^2 & 0<x\le 1 \\ 3x-2 & x>1 \end{cases}$$





You will have to remember to graph the point (0, 1). What line does that point come from?
The secret to graphing a piece-wise function is to graph each individual part separately for only the *x*-values for which it is defined. Sometimes the pieces connect; sometimes they don't. Be very careful at those points.

It is your responsibility to check all endpoint of sections of the graph to know whether or not there are open circles or closed circles. Also, for this particular function, the graph looks like there is a hole at x = 1; however, from the function you know that this point (1, 1) is filled in. In your calculator manual, there are other ways of graphing a piecewise defined function. Check these ways out; they are really better than the above method.
Please remember your interval notation that is reviewed as follows:

$[a,b] \to a \le x \le b$

$(a,b] \to a < x \le b$

$[a,b) \to a \le x < b$

$(a,b) \to a < x < b$

$[a,\infty) \to x \ge a$

$(a,\infty) \to x > a$

$(-\infty,a] \to x \le a$

$(-\infty,a) \to x < a$

$(-\infty,\infty) \to$ all real numbers

Facts About Absolute Value - Definition: $f(x)=|x|=\begin{cases} x & \text{if} \quad x\ge 0 \\ -x & \text{if} \quad x<0 \end{cases}$

Less Than

1. $|x|<0$ - no solution - ϕ
2. $|x|\le 0$ - one solution - $x=0$
3. $|x|<$ a negative number - no solution - ϕ
4. $|x|\le$ a negative number - no solution - ϕ
5. $|x|<$ a (a positive number) – conjunction - $-a<x<a$
6. $|x|\le$ a (a positive number) – conjunction - $-a\le x\le a$

Greater Than

1. $|x| > 0$ - everything except $x = 0$
2. $|x| \geq 0$ - everything – all reals
3. $|x| >$ a negative number – everything – all reals
4. $|x| > a$ (a positive number) – disjunction - $x < -a$ or $x > a$
5. $|x| \geq a$ (a positive number) – disjunction - $x \leq -a$ or $x \geq a$

Equal To

1. $|x| = a$ (a positive number) – 2 solutions - $x = a$ or $x = -a$
2. $|x| = 0$ - 1 solution - $x = 0$

Transformations - THE 12 FRIENDLY TRANSFORMATIONS

Each year in calculus we do a lot of work with transformations of graphs. AP Calculus teachers make the implicit (though dangerous) assumption that each student entering their course will have quick mental images of several functions and the ability to do transformations of those functions. This assignment is meant to ensure that these assumptions are indeed true.

Transformations of functions are the processes of shifting, stretching, compressing, reflecting, etc., existing functions to obtain new functions. There are:

(1) translations (shifting of original function either horizontally or vertically, keeping the shape the same),
(2) transformations (stretching or compressing the original function either horizontally or vertically which changes the shape of the original function),
(3) reflections (a.k.a. flips) (reflecting the original function over the *x*- or *y*-axis), and
(4) absolute value (alters the shape of the graph considerably).

One purpose of transformations is to make graphing easier. If you know the general shape of the original function, then you will be able to quickly shift, stretch, compress, reflect, etc., the original function to get the function that you desire. The 12 friendly transformations are:

<u>Translations</u>

Vertical shifts	$f(x)+a$	shifts graph up *a* units
	$f(x)-a$	shifts graph down *a* units
Horizontal shifts	$f(x+a)$	shifts graph left *a* units
	$f(x-a)$	shifts graph right *a* units

<u>Stretches and Compressions</u>

Vertical stretch	$a \cdot f(x)$	steeper, x-intercepts same
Horizontal compression	$f(a \cdot x)$	squinched, y-intercepts same
Vertical compression	$\dfrac{f(x)}{a}$	wider, x-intercepts same
Horizontal stretch	$f(\frac{x}{a})$	stretched, y-intercepts same

<u>Reflections</u>

$f(-x)$	reflected over y-axis

$-f(x)$ reflected over x-axis

Absolute Value

$f(|x|)$ ignore negative x's, keep graph of positive x's the same, reflect that graph over y-axis

$|f(x)|$ keep positive y's the same, flip negative y's up over x-axis

Combinations and Compositions

Combinations - Two functions *f(x)* and *g(x)* can be combined to form new functions, such as $f + g$, $f - g$, fg, and f/g, the same way that we add, subtract, multiply, and divide real numbers. The domain of these functions is the intersection of the domains of the individual functions and for *f/g*, only the values for which $g(x) \neq 0$.

Compositions - This is another way of combining functions. This process involves two functions where you find the first function of the initial value, and then find the second function of the answer to the first function. In mathematical symbols, the process looks like this: $(f \circ g)(x) = f(g(x))$. Remember, this notation $f \circ g$ means that the function *g* is performed first and then *f* is performed on the answer to *g*. The domain of the composition of two functions *f* and *g* is the set of all *x* in the domain of *g* such that *g(x)* is in the domain of *f*.

Inverse Functions

Inverse functions have a special property. If *f* and *g* are functions, then $(f \circ g)(x) = (g \circ f)(x) = x$, or $f(g(x)) = g(f(x)) = x$. Inverse functions have several characteristics unique to them:

(1) If (x, y) is on the original function, then (y, x) is on the inverse function.

(2) Inverse functions are symmetrical about the line $y = x$.

(3) You can find the inverse function of an original function by switching the *x* and *y* in the original function, and solving for *y*. Sometimes this process is difficult, and you cannot easily solve for *y*.

(4) The notation used for inverse functions is $f^{-1}(x)$.

For example, let $f(x) = 3x - 5$. Then by substituting *y* for *f(x)* in the equation, switching the *x* and *y* in the equation and then solving for *y*, you will find the new inverse function to be $f^{-1}(x) = \dfrac{x+5}{3}$. The graphs of these functions are as follows with *f(x)* graphed in bold: