AP® Precalculus

PROPOSED COURSE FRAMEWORK

November 2022
Preview

Equity and Access

College Board strongly encourages educators to make equitable access a guiding principle for their AP programs by giving all willing and academically prepared students the opportunity to participate in AP. We encourage the elimination of barriers that restrict access to AP for students from ethnic, racial, and socioeconomic groups that have been traditionally underrepresented. Schools should make every effort to ensure their AP classes reflect the diversity of their student population. College Board also believes that all students should have access to academically challenging coursework before they enroll in AP classes, which can prepare them for AP success. It is only through a commitment to equitable preparation and access that true equity and excellence can be achieved.

Contents

Acknowledgements

College Board would like to acknowledge the following individuals for their assistance with and contributions to the development of this course. All individuals' affiliations were current at the time of contribution.

ADVISORY BOARD

David Bressoud, *Macalester College*, Saint Paul, MN

Marilyn Carlson, *Arizona State University*, Tempe, AZ

Stephen Davis, *Davidson College*, Davidson, NC

Kristin Frank, *Towson University*, Towson, MD

Adrian Mims, *The Calculus Project*, Brookline, MA

Roberto Pelayo, *University of California – Irvine*, Irvine, CA

CONTENT WRITING TEAM

Grace Cook, *Bloomfield College*, Bloomfield, NJ

Tracey Etheredge-Alford, *Barack Obama Male Leadership Academy*, Dallas, TX

Rebecca George, *Georgia Institute of Technology*, Atlanta, GA

Jerome Heaven, *Indiana Institute of Technology*, Fort Wayne, IN

Roberto Pelayo, *University of California – Irvine*, Irvine, CA

Paul Rodriguez, *Troy High School*, Fullerton, CA

SPECIAL CONTENT REVIEWER

Michael Boardman, *Pacific University*, Forest Grove, OR

PACING AND IMPLEMENTATION REVIEWERS

Julio Avasan, *Olympian High School*, Chula Vista, CA

Vicki Carter, *West Florence High School*, Florence, SC

Virge Cornelius, *Lafayette County High School*, Oxford, MS

Tracey Etheredge-Alford, *Barack Obama Male Leadership Academy,* Dallas, TX

Eliel Gonzalez, *East Longmeadow High School*, East Longmeadow, MA

Mark Howell, *Gonzaga College High School*, Washington, DC

Jon Kawamura, *West Salem High School*, Salem, OR

Christina Martin, *Faith West Academy*, Katy, TX

Brendan Murphy, *John Bapst High School*, Bangor, ME

Andrew Oberlies, *Sequoyah High School*, Canton, GA

Melissa Walton, *Metamora Township High School*, Metamora, IL

COLLEGE BOARD STAFF

Jason VanBilliard, *Senior Director, AP Precalculus*

Natasha Vasavada, *Executive Director, New Course Development*

About AP

College Board's Advanced Placement® Program (AP®) enables willing and academically prepared students to pursue college-level studies—with the opportunity to earn college credit, advanced placement, or both—while still in high school. Through AP courses in 38 subjects, each culminating in a challenging exam, students learn to think critically, construct solid arguments, and see many sides of an issue—skills that prepare them for college and beyond. Taking AP courses demonstrates to college admission officers that students have sought the most challenging curriculum available to them, and research indicates that students who score a 3 or higher on an AP Exam typically experience greater academic success in college and are more likely to earn a college degree than non-AP students. Each AP teacher's syllabus is evaluated and approved by faculty from some of the nation's leading colleges and universities, and AP Exams are developed and scored by college faculty and experienced AP teachers. Most four-year colleges and universities in the United States grant credit, advanced placement, or both on the basis of successful AP Exam scores; more than 3,300 institutions worldwide annually receive AP scores.

AP Course Development

In an ongoing effort to maintain alignment with best practices in college-level learning, AP courses and exams emphasize challenging, research-based curricula aligned with higher education expectations.

Individual teachers are responsible for designing their own curriculum for AP courses and selecting appropriate college-level readings, assignments, and resources. This course framework document presents the content and skills that are the focus of the corresponding college course and that appear on the AP Exam. It also organizes the content and skills into a series of units that represent a sequence found in widely adopted college textbooks and that many AP teachers have told us they follow in order to focus their instruction. The intention of this publication is to respect teachers' time and expertise by providing a roadmap that they can modify and adapt to their local priorities and preferences. Moreover, by organizing the AP course content and skills into units, the AP Program is able to provide teachers and students with formative assessments—Progress Checks—that teachers can assign throughout the year to measure students' progress as they acquire content knowledge and develop skills.

Enrolling Students: Equity and Access

College Board strongly encourages educators to make equitable access a guiding principle for their AP programs by giving all willing and academically prepared students the opportunity to participate in AP. We encourage the elimination of barriers that restrict access to AP for students from ethnic, racial, and socioeconomic groups that have been traditionally underserved. College Board also believes that all students should have access to academically challenging coursework before they enroll in AP classes, which can prepare them for AP success. It is only through a commitment to equitable preparation and access that true equity and excellence can be achieved.

Offering AP Courses: The AP Course Audit

The AP Program unequivocally supports the principle that each school implements its own curriculum that will enable students to develop the content understandings and skills described in the course framework.

While the unit sequence presented in this publication is optional, the AP Program does have a short list of curricular and resource requirements that must be fulfilled before a school can label a course "Advanced Placement" or "AP." Schools wishing to offer AP courses must participate in the AP Course Audit, a process through which AP teachers' course materials are reviewed by college faculty. The AP Course Audit was created to provide teachers and administrators with clear guidelines on curricular and resource requirements for AP courses and to help colleges and universities validate courses marked "AP" on students' transcripts. This process ensures that AP teachers' courses meet or exceed the curricular and resource expectations that college and secondary school faculty have established for college-level courses.

The AP Course Audit form is submitted by the AP teacher and the school principal (or designated administrator) to confirm awareness and understanding of the curricular and resource requirements. A syllabus or course outline, detailing how course requirements are met, is submitted by the AP teacher for review by college faculty.

Please visit **collegeboard.org/apcourseaudit** for more information to support the preparation and submission of materials for the AP Course Audit.

How the AP Program Is Developed

The scope of content for an AP course and exam is derived from an analysis of hundreds of syllabi and course offerings of colleges and universities. Using this research and data, a committee of college faculty and expert AP teachers work within the scope of the corresponding college course to articulate what students should know and be able to do upon the completion of the AP course. The resulting course framework serves as a blueprint of the content and skills that can appear on an AP Exam.

The AP Test Development Committees are responsible for developing each AP Exam, ensuring the exam questions are aligned to the course framework. The AP Exam development process is a multiyear endeavor; all AP Exams undergo extensive review, revision, and analysis to ensure that questions are accurate, fair, and valid and that there is an appropriate spread of difficulty across the questions.

Committee members are selected to represent a variety of perspectives and institutions (public and private, small and large schools and colleges) and a range of gender, racial/ethnic, and regional groups. A list of each subject's current AP Test Development Committee members is available on **apcentral.collegeboard.org**.

Throughout AP course and exam development, College Board gathers feedback from various stakeholders in both secondary schools and higher education institutions. This feedback is carefully considered to ensure that AP courses and exams are able to provide students with a college-level learning experience and the opportunity to demonstrate their qualifications for advanced placement or college credit.

How AP Exams Are Scored

The exam scoring process, like the course and exam development process, relies on the expertise of both AP teachers and college faculty. While multiple-choice questions are scored by machine, the free-response questions and through-course performance assessments, as applicable, are scored by thousands of college faculty and expert AP teachers. Most are scored at the annual AP Reading, while a small portion are scored online. All AP readers are thoroughly trained, and their work is monitored throughout the Reading for fairness and consistency. In each subject, a highly respected college faculty member serves as Chief Faculty Consultant and, with the help of AP readers in leadership positions, maintains the accuracy of the scoring standards. Scores on the free-response questions and performance assessments are weighted and combined with the results of the computer-scored multiple-choice questions, and this raw score is converted into a composite AP score on a 1–5 scale.

AP Exams are **not** norm-referenced or graded on a curve. Instead, they are criterion-referenced, which means that every student who meets the criteria for an AP score of 2, 3, 4, or 5 will receive that score, no matter how many students that is. The criteria for the number of points students must earn on the AP Exam to receive scores of 3, 4, or 5—the scores that research consistently validates for credit and placement purposes—include:

- The number of points successful college students earn when their professors administer AP Exam questions to them
- The number of points researchers have found to be predictive that an AP student will succeed when placed into a subsequent, higher-level college course
- Achievement-level descriptions formulated by college faculty who review each AP Exam question

AP PRECALCULUS

Course Framework

Introduction

AP Precalculus centers on functions modeling dynamic phenomena. This research-based exploration of functions is designed to better prepare students for college-level calculus and provide grounding for other mathematics and science courses. In this course, students study a broad spectrum of function types that are foundational for careers in mathematics, physics, biology, health science, social science, and data science. Furthermore, as AP Precalculus may be the last mathematics course of a student's secondary education, the course is structured to provide a coherent capstone experience and is not exclusively focused on preparation for future courses.

During this course, students acquire and apply mathematical tools in real-world modeling situations in preparation for using these tools in college-level calculus. Modeling, a central instructional theme for the course, helps students come to a deeper understanding of each function type. By examining scenarios, conditions, and data sets, as well as determining and validating an appropriate function model, students develop a greater comprehension of the nature and behavior of the function itself. The formal study of a function type through multiple representations (e.g., graphical, numerical, verbal, analytical), coupled with the application of the function type to a variety of contexts, provides students with a rich study of precalculus.

Throughout this course, students develop and hone symbolic manipulation skills needed for future mathematics courses. They also solve equations and manipulate expressions for the many function types throughout the course. Students also learn that functions and their compositions, inverses, and transformations are understood through graphical, numerical, verbal, and analytical representations, which reveal different attributes of the functions and are useful for solving problems in mathematical and applied contexts. In turn, the skills learned in this course are widely applicable in a variety of future courses that involve quantitative reasoning.

AP Precalculus fosters the development of a deep conceptual understanding of functions. Students learn that a function is a mathematical relation that maps a set of input values—the domain—to a set of output values—the range—such that each input value is uniquely mapped to an output value. At various points and over various intervals, a function takes on characteristics that can be classified with varying levels of precision and justification, depending on the function representation and available mathematical tools. Furthermore, a function can be classified as part of a function family based on the way in which values of different variables change simultaneously.

Research indicates that deep understanding of functions and their graphs as embodying dynamic covariation of quantities best supports student preparation for calculus. With each function type, students develop and validate function models based on the characteristics of a bivariate data set, characteristics of covarying quantities and their relative rates of change, or a set of characteristics such as zeros, asymptotes, and extrema. These models are used to interpolate, extrapolate, and interpret information with varying degrees of accuracy for a given context or data set. Additionally, students also learn that every model is subject to assumptions and limitations related to the context. As a result of examining functions from many perspectives, students develop a conceptual understanding not only of specific function types but also of functions in general. This type of understanding helps students to engage with both familiar and novel contexts.

Unit Outline

Unit 1: Polynomial and Rational Functions

Unit 2: Exponential and Logarithmic Functions

Unit 3: Trigonometric and Polar Functions

Unit 4: Functions Involving Parameters, Vectors, and Matrices

Unit Notes

Each unit includes these features:

- Exploration, analysis, and application of **new function types**.
- Deep development of a **key function concept** applicable across function types such as transformations, compositions, and inverses.
- Examination of **how variables change relative to each other** for each of the function types.
- Use of each function type to **model contexts and data sets**.
- Rigorous application of the **algebraic skills** needed to engage with each function type.

Technology Notes

Technology should be used throughout the course as a tool to explore concepts. In AP Precalculus, students should specifically practice using technology to do the following:

- Perform calculations (e.g., exponents, roots, trigonometric values, logarithms)
- Graph functions and analyze graphs
- Generate a table of values for a function
- Find real zeros of functions
- Find points of intersection of graphs of functions
- Find minima/maxima of functions
- Find numerical solutions to equations in one variable
- Find regressions equations to model data
- Perform matrix operations (e.g., multiplication, finding inverses)

However, it is important to note that technology should not replace the development of symbolic manipulation skills. When algebraic expressions and equations are accessible with precalculus-level algebraic manipulation, students are expected to find zeros, solve equations, and calculate values without the help of technology. Most of the AP Exam will need to be completed without the use of technology. However, selected questions will require students to use a graphing calculator to complete the tasks delineated above.

Expected Prior Knowledge and Skills

- Proficiency with linear functions
- Proficiency in polynomial addition and multiplication
- Proficiency in factoring quadratic trinomials
- Proficiency in using the quadratic formula
- Proficiency in solving right triangle problems involving trigonometry
- Proficiency in solving linear and quadratic equations and inequalities
- Proficiency in algebraic manipulation of linear equations and expressions
- Proficiency in solving systems of equations in two and three variables
- Familiarity with piecewise-defined functions
- Familiarity with exponential functions and rules for exponents
- Familiarity with radicals (e.g., square roots, cube roots)
- Familiarity with complex numbers

Course Framework Components

Overview

This course framework provides a clear and detailed description of the course requirements necessary for student success. The framework specifies what students should know, be able to do, and understand to qualify for college credit or placement.

The course framework includes two essential components:

MATHEMATICAL PRACTICES

The mathematical practices are central to the study and practice of precalculus. Students should develop and apply the described skills on a regular basis over the span of the course.

COURSE CONTENT

The course content is organized into units of study that provide a suggested sequence for the course. These units comprise the content and conceptual understandings that colleges and universities typically expect students to master to qualify for college credit and/or placement.

COURSE FRAMEWORK CONVENTIONS:

Common language usage (e.g., “area of a triangle”) replaces precise mathematical phrasing (e.g., “area of the interior of a triangle”) in the following instances:

- When the framework refers to modeling a data set, it is referring to a bivariate data set.
- When the framework refers to modeling a context or phenomenon, it is referring to two aspects of the context or phenomena.
- When the framework refers to the sine, cosine, and so on of an angle, it is referring to the sine, cosine, and so on of the measure of the angle.

Mathematical Practices

The eight distinct skills are associated with three mathematical practices. Students should build and master these skills throughout the course. While many different skills can be applied to any one content topic, the framework supplies skill focus recommendations for each topic to help assure skill distribution throughout the course.

Practice 1	Practice 2	Practice 3
Procedural and Symbolic Fluency Algebraically manipulate functions, equations, and expressions.	**Multiple Representations** Translate mathematical information between representations.	**Communication and Reasoning** Communicate with precise language, and provide rationales for conclusions.
Skill 1.A: Solve equations and inequalities represented analytically, with and without technology. **Skill 1.B: Express** functions, equations, or expressions in analytically **equivalent forms** that are useful in a given mathematical or applied context. **Skill 1.C: Construct new functions**, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.	**Skill 2.A: Identify information from** graphical, numerical, analytical, and verbal **representations** to answer a question or construct a model, with and without technology. **Skill 2.B: Construct equivalent** graphical, numerical, analytical, and verbal **representations** of functions that are useful in a given mathematical or applied context, with and without technology.	**Skill 3.A: Describe** the **characteristics** of a function with varying levels of precision, depending on the function representation and available mathematical tools. **Skill 3.B: Apply** numerical **results** in a given mathematical or applied context. **Skill 3.C: Support conclusions** or choices with a logical rationale or appropriate data.

Course at a Glance

UNITS 1–3: AP Precalculus Exam Topics (required for college calculus placement)

Unit 1	Polynomial and Rational Functions
1.1	Change in Tandem
1.2	Rates of Change
1.3	Rates of Change in Linear and Quadratic Functions
1.4	Polynomial Functions and Rates of Change
1.5	Polynomial Functions and Complex Zeros
1.6	Polynomial Functions and End Behavior
1.7	Rational Functions and End Behavior
1.8	Rational Functions and Zeros
1.9	Rational Functions and Vertical Asymptotes
1.10	Rational Functions and Holes
1.11	Equivalent Representations of Polynomial and Rational Expressions
1.12	Transformations of Functions
1.13	Function Model Selection and Assumption Articulation
1.14	Function Model Construction and Application

Unit 2	Exponential and Logarithmic Functions
2.1	Change in Arithmetic and Geometric Sequences
2.2	Change in Linear and Exponential Functions
2.3	Exponential Functions
2.4	Exponential Function Manipulation
2.5	Exponential Function Context and Data Modeling
2.6	Competing Function Model Validation
2.7	Composition of Functions
2.8	Inverse Functions
2.9	Logarithmic Expressions
2.10	Inverses of Exponential Functions
2.11	Logarithmic Functions
2.12	Logarithmic Function Manipulation
2.13	Exponential and Logarithmic Equations and Inequalities
2.14	Logarithmic Function Context and Data Modeling
2.15	Semi-log Plots

Unit 3	Trigonometric and Polar Functions
3.1	Periodic Phenomena
3.2	Sine, Cosine, and Tangent
3.3	Sine and Cosine Function Values
3.4	Sine and Cosine Function Graphs
3.5	Sinusoidal Functions
3.6	Sinusoidal Function Transformations
3.7	Sinusoidal Function Context and Data Modeling
3.8	The Tangent Function
3.9	Inverse Trigonometric Functions
3.10	Trigonometric Equations and Inequalities
3.11	The Secant, Cosecant, and Cotangent Functions
3.12	Equivalent Representations of Trigonometric Functions
3.13	Trigonometry and Polar Coordinates
3.14	Polar Function Graphs
3.15	Rates of Change in Polar Functions

UNIT 4: Additional Topics Available to Schools (not included on AP Precalculus Exam)

Unit 4	Functions Involving Parameters, Vectors, and Matrices
4.1	Parametric Functions
4.2	Parametric Functions Modeling Planar Motion
4.3	Parametric Functions and Rates of Change
4.4	Parametrically Defined Circles and Lines
4.5	Implicitly Defined Functions
4.6	Conic Sections
4.7	Parametrization of Implicitly Defined Functions
4.8	Vectors
4.9	Vector-Valued Functions
4.10	Matrices
4.11	The Inverse and Determinant of a Matrix
4.12	Linear Transformations and Matrices
4.13	Matrices as Functions
4.14	Matrices Modeling Contexts

UNIT 1:

Polynomial and Rational Functions

Unit at a Glance

Topic #	Topic Title	Suggested Skill Focus
1.1	Change in Tandem	2.B, 3.A
1.2	Rates of Change	2.A, 3.A
1.3	Rates of Change in Linear and Quadratic Functions	3.B, 3.C
1.4	Polynomial Functions and Rates of Change	2.A, 3.A
1.5	Polynomial Functions and Complex Zeros	1.B, 2.B
1.6	Polynomial Functions and End Behavior	3.A
1.7	Rational Functions and End Behavior	1.B, 3.A
1.8	Rational Functions and Zeros	1.A
1.9	Rational Functions and Vertical Asymptotes	2.A
1.10	Rational Functions and Holes	3.C
1.11	Equivalent Representations of Polynomial and Rational Expressions	1.B, 3.B
1.12	Transformations of Functions	1.C, 3.A
1.13	Function Model Selection and Assumption Articulation	2.A, 3.C
1.14	Function Model Construction and Application	1.C, 3.B

TOPIC 1.1

Change in Tandem

Skills Focus: 2.B, 3.A

LEARNING OBJECTIVES

1.1.A Describe how the input and output values of a function vary together by comparing function values.

ESSENTIAL KNOWLEDGE

- **1.1.A.1** A *function* is a mathematical relation that maps a set of input values to a set of output values such that each input value is mapped to exactly one output value. The set of input values is called the *domain* of the function, and the set of output values is called the *range* of the function. The variable representing input values is called the *independent variable*, and the variable representing output values is called the *dependent variable*.
- **1.1.A.2** The input and output values of a function vary in tandem according to the *function rule*, which can be expressed graphically, numerically, analytically, or verbally.
- **1.1.A.3** A function is increasing over an interval of its domain if, as the input values increase, the output values always increase. That is, for all a and b in the interval, if $a < b$, then $f(a) < f(b)$.
- **1.1.A.4** A function is decreasing over an interval of its domain if, as the input values increase, the output values always decrease. That is, for all a and b in the interval, if $a < b$, then $f(a) > f(b)$.

LEARNING OBJECTIVES

1.1.B Construct a graph representing two quantities that vary with respect to each other in a contextual scenario.

ESSENTIAL KNOWLEDGE

- **1.1.B.1** The graph of a function displays a set of input-output pairs and shows how the values of the function's input and output values vary.
- **1.1.B.2** A verbal description of the way aspects of phenomena change together can be the basis for constructing a graph.
- **1.1.B.3** The graph of a function is *concave up* on intervals in which the rate of change is increasing.
- **1.1.B.4** The graph of a function is *concave down* on intervals in which the rate of change is decreasing.
- **1.1.B.5** The graph intersects the *x*-axis when the output value is zero. The corresponding input values are said to be *zeros of the function*.

TOPIC 1.2

Rates of Change

Skills Focus: 2.A, 3.A

LEARNING OBJECTIVES	ESSENTIAL KNOWLEDGE
1.2.A Compare the rates of change at two points using average rates of change near the points.	▪ **1.2.A.1** The average rate of change of a function over an interval of the function's domain is the constant rate of change that yields the same change in the output values as the function yielded on that interval of the function's domain. It is the ratio of the change in the output values to the change in input values over that interval. ▪ **1.2.A.2** The rate of change of a function at a point quantifies the rate at which output values would change were the input values to change at that point. The rate of change at a point can be approximated by the average rates of change of the function over small intervals containing the point, if such values exist. ▪ **1.2.A.3** The rates of change at two points can be compared using average rate of change approximations over sufficiently small intervals containing each point, if such values exist.
1.2.B Describe how two quantities vary together at different points and over different intervals of a function.	▪ **1.2.B.1** Rates of change quantify how two quantities vary together. ▪ **1.2.B.2** A positive rate of change indicates that as one quantity increases or decreases, the other quantity does the same. ▪ **1.2.B.3** A negative rate of change indicates that as one quantity increases, the other decreases.

TOPIC 1.3

Rates of Change in Linear and Quadratic Functions

Skills Focus: 3.B, 3.C

LEARNING OBJECTIVES	ESSENTIAL KNOWLEDGE
1.3.A Determine the average rates of change for linear and quadratic sequences and functions.	▪ **1.3.A.1** Over any length input-value interval, the average rate of change for a linear function is constant. ▪ **1.3.A.2** For consecutive equal-length input-value intervals, the average rate of change of a quadratic function can be given by a linear function. ▪ **1.3.A.3** The average rate of change over the closed interval $\left[a, b\right]$ is the slope of the secant line from the point $\left(a, f\left(a\right)\right)$ to $\left(b, f\left(b\right)\right)$.
1.3.B Determine the change of average rates of change for linear and quadratic functions.	▪ **1.3.B.1** Because the average rate of change of a linear function over any length input-value interval is constant, the rate of change of the average rates of change of a linear function is zero. ▪ **1.3.B.2** Because the average rate of change of a quadratic function over consecutive equal-length input-value intervals can be given by a linear function, the rate of change of the average rates of change of a quadratic function is constant. ▪ **1.3.B.3** When the average rate of change over equal-length input-value intervals is increasing for all small-length intervals, the graph of the function is concave up. When the average rate of change over equal-length input-value intervals is decreasing for all small-length intervals, the graph of the function is concave down.

TOPIC 1.4

Polynomial Functions and Rates of Change

Skills Focus: 2.A, 3.A

LEARNING OBJECTIVES

1.4.A Identify key characteristics of polynomial functions related to rates of change.

ESSENTIAL KNOWLEDGE

- **1.4.A.1** A nonconstant polynomial function of x is any function representation that is equivalent to the analytical form $p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0$, where n is a positive integer, a_i is a real number for each i from 1 to n, and a_n is nonzero. The polynomial has degree n, the leading term is $a_n x^n$, and the leading coefficient is a_n. A constant is also a polynomial function of degree zero.
- **1.4.A.2** Where a polynomial function switches between increasing and decreasing, or at the included endpoint of a polynomial with a restricted domain, the polynomial function will have a *local*, or *relative*, maximum or minimum output value. Of all local maxima, the greatest is called the *global*, or *absolute*, maximum. Likewise, the least of all local minima is called the global, or absolute, minimum.
- **1.4.A.3** Between every two distinct real zeros of a nonconstant polynomial function, there must be at least one input value corresponding to a local maximum or local minimum.
- **1.4.A.4** Polynomial functions of an even degree will have either a global maximum or a global minimum.
- **1.4.A.5** *Points of inflection* of a polynomial function occur at input values where the rate of change of the function changes from increasing to decreasing or from decreasing to increasing. This occurs where the graph of a polynomial function changes from concave up to concave down or from concave down to concave up.

TOPIC 1.5

Polynomial Functions and Complex Zeros

Skills Focus: 1.B, 2.B

LEARNING OBJECTIVES

1.5.A Identify key characteristics of a polynomial function related to its zeros when suitable factorizations are available or with technology.

ESSENTIAL KNOWLEDGE

- **1.5.A.1** If a is a complex number and $p(a)=0$, then a is called a *zero* of p, or a *root* of the polynomial function p. If a is a real number, then $(x-a)$ is a linear factor of p if and only if a is a zero of p.
- **1.5.A.2** If a linear factor $(x-a)$ is repeated n times, the corresponding zero of the polynomial function has a *multiplicity* n. A polynomial of *degree* n has exactly n complex zeros when counting multiplicities.
- **1.5.A.3** If a is a real root of a polynomial function p, then the graph of $y=p(x)$ has an *x*-intercept at the point $(a,0)$. Consequently, real zeros of a polynomial can be endpoints for intervals satisfying polynomial inequalities.
- **1.5.A.4** If $a+bi$ is a non-real zero of a polynomial p, then its *conjugate* $a-bi$ is also a zero of p.
- **1.5.A.5** If the real zero, a, of a polynomial function has even multiplicity, then the signs of the output values are the same for input values near $x=a$. For these polynomials, the graph will be tangent to the *x*-axis at $x=a$.
- **1.5.A.6** The degree of a polynomial function can be found by examining the successive differences of the output values over equal-interval input values. The degree of the polynomial function is equal to the least value n for which the successive nth differences are constant.

LEARNING OBJECTIVES	ESSENTIAL KNOWLEDGE
1.5.B Determine if a polynomial is even or odd.	▪ **1.5.B.1** An *even* function is graphically symmetric over the line $x=0$ and analytically has the property $f(-x)=f(x)$. If n is even, then a polynomial of the form $p(x)=a_n x^n$, where $n \geq 1$ and $a_n \neq 0$, is an even function. ▪ **1.5.B.2** An *odd* function is graphically symmetric about the point $(0,0)$ and analytically has the property $f(-x)=-f(x)$. If n is odd, then a polynomial of the form $p(x)=a_n x^n$, where $n \geq 1$ and $a_n \neq 0$, is an odd function.

TOPIC 1.6

Polynomial Functions and End Behavior

Skills Focus: 3.A

LEARNING OBJECTIVES

1.6.A Describe end behaviors of polynomial functions.

ESSENTIAL KNOWLEDGE

- **1.6.A.1** As input values of a nonconstant polynomial function increase without bound, the output values will either increase or decrease without bound. The corresponding mathematical notation is $\lim_{x\to\infty} p(x)=\infty$ or $\lim_{x\to\infty} p(x)=-\infty$.
- **1.6.A.2** As input values of a nonconstant polynomial function decrease without bound, the output values will either increase or decrease without bound. The corresponding mathematical notation is $\lim_{x\to-\infty} p(x)=\infty$ or $\lim_{x\to-\infty} p(x)=-\infty$.
- **1.6.A.3** The degree and sign of the leading term of a polynomial determines the end behavior of the polynomial function, because as the input values increase or decrease without bound, the values of the leading term dominate the values of all lower-degree terms.

TOPIC 1.7

Rational Functions and End Behavior

Skills Focus: 1.B, 3.A

LEARNING OBJECTIVES

1.7.A Describe end behaviors of rational functions.

ESSENTIAL KNOWLEDGE

- **1.7.A.1** A rational function is analytically represented as a quotient of two polynomial functions and gives a measure of the relative size of the polynomial function in the numerator compared to the polynomial function in the denominator for each value in the rational function's domain.
- **1.7.A.2** The end behavior of a rational function will be affected most by the polynomial with the greater degree, as its values will dominate the values of the rational function for input values of large magnitude. For input values of large magnitude, a polynomial is dominated by its leading term. Therefore, the end behavior of a rational function can be understood by examining the corresponding quotient of the leading terms.
- **1.7.A.3** If the polynomial in the numerator dominates the polynomial in the denominator for input values of large magnitude, then the quotient of the leading terms is a nonconstant polynomial, and the original rational function has the end behavior of that polynomial. If that polynomial is linear, then the rational function has a slant asymptote parallel to the graph of the line.
- **1.7.A.4** If neither polynomial in a rational function dominates the other for input values of large magnitude, then the quotient of the leading terms is a constant, and that constant indicates the location of a horizontal asymptote of the original rational function.
- **1.7.A.5** If the polynomial in the denominator dominates the polynomial in the numerator for input values of large magnitude, then the quotient of the leading terms is a rational function with a constant in the numerator and nonconstant polynomial in the denominator, and the original rational function has a horizontal asymptote at $y=0$.
- **1.7.A.6** When the graph of a rational function has a horizontal asymptote $y=b$, where b is a constant, the output values of the rational function get arbitrarily close to b and stay arbitrarily close to b as input values increase or decrease without bound. The corresponding mathematical notation is $\lim_{x\to\infty} r(x)=b$ or $\lim_{x\to-\infty} r(x)=b$.

TOPIC 1.8

Rational Functions and Zeros

Skills Focus: 1.A

LEARNING OBJECTIVES	ESSENTIAL KNOWLEDGE
1.8.A Determine the zeros of rational functions.	▪ **1.8.A.1** The real zeros of a rational function correspond to the real zeros of the numerator for such values in its domain. ▪ **1.8.A.2** The real zeros of both polynomials of a rational function are endpoints or asymptotes for intervals satisfying the rational function inequalities $r(x) \geq 0$ or $r(x) \leq 0$.

TOPIC 1.9

Rational Functions and Vertical Asymptotes

Skills Focus: 2.A

LEARNING OBJECTIVES

1.9.A Determine vertical asymptotes of rational functions.

ESSENTIAL KNOWLEDGE

- **1.9.A.1** If the value a is a real zero of the polynomial in the denominator of a rational function and is not also a real zero of the polynomial in the numerator, then the graph of the rational function has a vertical asymptote at $x = a$. Furthermore, a vertical asymptote also occurs at $x = a$ if the multiplicity of a as a real zero in the denominator is greater than its multiplicity as a real zero in the numerator.
- **1.9.A.2** Near a vertical asymptote, $x = a$, of a rational function, the values of the polynomial in the denominator are arbitrarily close to zero, so the values of the rational function increase or decrease without bound. The corresponding mathematical notation is $\lim_{x \to a^+} r(x) = \infty$ or $\lim_{x \to a^+} r(x) = -\infty$ for input values near a and greater than a, and $\lim_{x \to a^-} r(x) = \infty$ or $\lim_{x \to a^-} r(x) = -\infty$ for input values near a and less than a.

TOPIC 1.10

Rational Functions and Holes

Skills Focus: 3.C

LEARNING OBJECTIVES

1.10.A Determine holes in graphs of rational functions.

ESSENTIAL KNOWLEDGE

- **1.10.A.1** If the multiplicity of a real zero in the numerator is greater than or equal to its multiplicity in the denominator, then the graph of the rational function has a hole at the corresponding input value.
- **1.10.A.2** If the graph of a rational function has a hole at $x = c$, then the location of the hole can be determined by examining the output values corresponding to input values sufficiently close to c. If input values sufficiently close to c correspond to output values arbitrarily close to L, then the hole is located at the point with coordinates (c, L). The corresponding mathematical notation is $\lim_{x \to c} r(x) = L$.

  It should be noted that $\lim_{x \to c^-} r(x) = \lim_{x \to c^+} r(x) = \lim_{x \to c} r(x) = L$.

TOPIC 1.11

Equivalent Representations of Polynomial and Rational Expressions

Skills Focus: 1.B, 3.B

LEARNING OBJECTIVES

1.11.A Rewrite polynomial and rational expressions in equivalent forms.

1.11.B Determine the quotient of two polynomials using long division.

1.11.C Rewrite the repeated product of binomials using the binomial theorem.

ESSENTIAL KNOWLEDGE

- **1.11.A.1** Because the factored form of a polynomial or rational function readily provides information about real zeros, it can reveal information about *x*-intercepts, asymptotes, holes, domain, and range.
- **1.11.A.2** The standard form of a polynomial or rational function can reveal information about end behavior of the function.
- **1.11.A.3** The information extracted from different analytic representations of the same polynomial or rational function can be used to answer questions in context.
- **1.11.B.1** Polynomial long division is an algebraic process similar to numerical long division involving a quotient and remainder. If the polynomial f is divided by the polynomial g, then f can be rewritten as $f(x)=g(x)q(x)+r(x)$, where q is the quotient, r is the remainder and the degree of r is less than the degree of g.
- **1.11.B.2** The result of polynomial long division is helpful in finding equations of slant asymptotes.
- **1.11.C.1** The binomial theorem utilizes the entries in a single row of Pascal's Triangle to more easily expand expressions of the form $(a+b)^n$, including polynomial functions of the form $p(x)=(x+c)^n$, where c is a constant.

TOPIC 1.12

Transformations of Functions

Skills Focus: 1.C, 3.A

LEARNING OBJECTIVES

1.12.A Construct a function that is an additive and/or multiplicative transformation of another function.

ESSENTIAL KNOWLEDGE

- **1.12.A.1** The function $g(x) = f(x) + k$ is an additive transformation of the function f that results in a vertical translation of the graph of f by k units.
- **1.12.A.2** The function $g(x) = f(x + h)$ is an additive transformation of the function f that results in a horizontal translation of the graph of f by $-h$ units.
- **1.12.A.3** The function $g(x) = af(x)$, where $a \neq 0$, is a multiplicative transformation of the function f that results in a vertical dilation of the graph of f by a factor of $|a|$. If $a < 0$, the transformation involves a reflection over the *x*-axis.
- **1.12.A.4** The function $g(x) = f(bx)$, where $b \neq 0$, is a multiplicative transformation of the function f that results in a horizontal dilation of the graph of f by a factor of $\left|\frac{1}{b}\right|$. If $b < 0$, the transformation involves a reflection over the *y*-axis.
- **1.12.A.5** Additive and multiplicative transformations can be combined, resulting in combinations of horizontal and vertical translations and dilations.
- **1.12.A.6** The domain and range of a function that is a transformation of a parent function may be different from those of the parent function.

TOPIC 1.13

Function Model Selection and Assumption Articulation

Skills Focus: 2.A, 3.C

LEARNING OBJECTIVES

1.13.A Identify an appropriate function type to construct a function model for a given scenario.

ESSENTIAL KNOWLEDGE

- **1.13.A.1** Linear functions model data sets or aspects of contextual scenarios that demonstrate roughly constant rates of change.
- **1.13.A.2** Quadratic functions model data sets or aspects of contextual scenarios that demonstrate roughly linear rates of change, or data sets that are roughly symmetric with a unique maximum or minimum value.
- **1.13.A.3** Geometric contexts involving area or two dimensions can often be modeled by quadratic functions. Geometric contexts involving volume or three dimensions can often be modeled by cubic functions.
- **1.13.A.4** Polynomial functions model data sets or contextual scenarios with multiple real zeros or multiple maxima or minima.
- **1.13.A.5** A polynomial function of degree n models data sets or contextual scenarios that demonstrate roughly constant nonzero $n\text{th}$ differences.
- **1.13.A.6** A polynomial function of degree n or less can be used to model a graph of $n+1$ points with distinct input values.
- **1.13.A.7** A piecewise-defined function consists of a set of functions defined over nonoverlapping domain intervals and is useful for modeling a data set or contextual scenario that demonstrates different characteristics over different intervals.

LEARNING OBJECTIVES

1.13.B Describe assumptions and restrictions related to building a function model.

ESSENTIAL KNOWLEDGE

- **1.13.B.1** A model may have underlying assumptions about what is consistent in the model.
- **1.13.B.2** A model may have underlying assumptions about how quantities change together.
- **1.13.B.3** A model may require domain restrictions based on mathematical clues, contextual clues, or extreme values in the data set.
- **1.13.B.4** A model may require range restrictions, such as rounding values, based on mathematical clues, contextual clues, or extreme values in the data set.

TOPIC 1.14

Function Model Construction and Application

Skills Focus: 1.C, 3.B

LEARNING OBJECTIVES	ESSENTIAL KNOWLEDGE
1.14.A Construct a linear, polynomial, or related piecewise-defined function model.	▪ **1.14.A.1** A model can be constructed based on restrictions identified in a mathematical or contextual scenario. ▪ **1.14.A.2** A model of a data set or a contextual scenario can be constructed using transformations of the parent function. ▪ **1.14.A.3** A model of a data set can be constructed using technology and regressions. ▪ **1.14.A.4** A piecewise-defined function model can be constructed through a combination of modeling techniques.
1.14.B Construct a rational function model based on a context.	▪ **1.14.B.1** Data sets and aspects of contextual scenarios involving quantities that are inversely proportional can often be modeled by rational functions. For example, the magnitudes of both gravitational force and electromagnetic force between objects are inversely proportional to the objects' squared distance.
1.14.C Apply a function model to answer questions about a data set or contextual scenario.	▪ **1.14.C.1** A model can be used to draw conclusions about the modeled data set or contextual scenario, including answering key questions and predicting values, rates of change, average rates of change, and changing rates of change. Appropriate units of measure should be extracted or inferred from the given context.

UNIT 2:

Exponential and Logarithmic Functions

Unit at a Glance

Topic #	Topic Title	Suggested Skill Focus
2.1	Change in Arithmetic and Geometric Sequences	1.B, 3.A
2.2	Change in Linear and Exponential Functions	1.C, 3.B
2.3	Exponential Functions	3.A
2.4	Exponential Function Manipulation	1.B, 3.A
2.5	Exponential Function Context and Data Modeling	1.C, 3.B
2.6	Competing Function Model Validation	2.A, 3.C
2.7	Composition of Functions	1.C, 2.B
2.8	Inverse Functions	1.A, 2.B
2.9	Logarithmic Expressions	1.B
2.10	Inverses of Exponential Functions	1.C, 2.B
2.11	Logarithmic Functions	3.A
2.12	Logarithmic Function Manipulation	1.B, 3.A
2.13	Exponential and Logarithmic Equations and Inequalities	1.A, 1.B, 1.C
2.14	Logarithmic Function Context and Data Modeling	1.C, 3.B
2.15	Semi-log Plots	2.B, 3.C

TOPIC 2.1

Change in Arithmetic and Geometric Sequences

Skills Focus: 1.B, 3.A

LEARNING OBJECTIVES

2.1.A Express arithmetic sequences found in mathematical and contextual scenarios as functions of the natural numbers.

2.1.B Express geometric sequences found in mathematical and contextual scenarios as functions of the natural numbers.

ESSENTIAL KNOWLEDGE

- **2.1.A.1** A *sequence* is a function from the whole numbers to the real numbers. Consequently, the graph of a sequence consists of discrete points instead of a curve.
- **2.1.A.2** Successive terms in an arithmetic sequence have a common difference, or constant rate of change.
- **2.1.A.3** The $n\text{th}$ term of an arithmetic sequence with a common difference d is denoted by a_n and is given by $a_n = a_0 + dn$, where a_0 is the initial value, or by $a_n = a_k + d(n-k)$, where a_k is the $k\text{th}$ term of the sequence.
- **2.1.B.1** Successive terms in a geometric sequence have a common ratio, or constant proportional change.
- **2.1.B.2** The $n\text{th}$ term of a geometric sequence with a common ratio r is denoted by g_n and is given by $g_n = g_0 r^n$, where g_0 is the initial value, or by $g_n = g_k r^{(n-k)}$, where g_k is the $k\text{th}$ term of the sequence.
- **2.1.B.3** Increasing arithmetic sequences increase equally with each step, whereas increasing geometric sequences increase by a larger amount with each successive step.

TOPIC 2.2

Change in Linear and Exponential Functions

Skills Focus: 1.C, 3.B

LEARNING OBJECTIVES

2.2.A Construct functions of the real numbers that are comparable to arithmetic and geometric sequences.

ESSENTIAL KNOWLEDGE

- **2.2.A.1** Linear functions of the form $f(x)=b+mx$ are similar to arithmetic sequences of the form $a_n=a_0+dn$, as both can be expressed as an initial value (b or a_0) plus repeated addition of a constant rate of change, the slope (m or d).
- **2.2.A.2** Similar to arithmetic sequences of the form $a_n=a_k+d(n-k)$, which are based on a known difference, d, and a $k\text{th}$ term, linear functions can be expressed in the form $f(x)=y_i+m(x-x_i)$ based on a known slope, m, and a point, (x_i, y_i).
- **2.2.A.3** Exponential functions of the form $f(x)=ab^x$ are similar to geometric sequences of the form $g_n=g_0r^n$, as both can be expressed as an initial value (a or g_0) times repeated multiplication by a constant proportion (b or r).
- **2.2.A.4** Similar to geometric sequences of the form $g_n=g_kr^{(n-k)}$, which are based on a known ratio, r, and a $k\text{th}$ term, exponential functions can be expressed in the form $f(x)=y_ir^{(x-x_i)}$ based on a known ratio, r, and a point, (x_i, y_i).
- **2.2.A.5** Sequences and their corresponding functions may have different domains.

LEARNING OBJECTIVES	ESSENTIAL KNOWLEDGE
2.2.B Describe similarities and differences between linear and exponential functions.	▪ **2.2.B.1** Over equal-length input-value intervals, if the output values of a function change at constant rate, then the function is linear; if the output values of a function change proportionally, then the function is exponential. ▪ **2.2.B.2** Linear functions of the form $f(x)=b+mx$ and exponential functions of the form $f(x)=ab^x$ can both be expressed analytically in terms of an initial value and a constant involved with change. However, linear functions are based on addition, while exponential functions are based on multiplication. ▪ **2.2.B.3** Arithmetic sequences, linear functions, geometric sequences, and exponential functions all have the property that they can be determined by two distinct sequence or function values.

TOPIC 2.3

Exponential Functions

Skills Focus: 3.A

LEARNING OBJECTIVES

2.3.A Identify key characteristics of exponential functions.

ESSENTIAL KNOWLEDGE

- **2.3.A.1** The general form of an exponential function is $f(x) = ab^x$, with the *initial value* a, where $a \neq 0$, and the *base* b, where $b > 0$, and $b \neq 1$. When $a > 0$ and $b > 1$, the exponential function is said to demonstrate *exponential growth*. When $a > 0$ and $0 < b < 1$, the exponential function is said to demonstrate *exponential decay*.
- **2.3.A.2** When the natural numbers are input values in an exponential function, the input value specifies the number of factors of the base to be applied to the function's initial value. The domain of an exponential function is all real numbers.
- **2.3.A.3** Because the output values of exponential functions in general form are proportional over equal-length input-value intervals, exponential functions are always increasing or always decreasing, and their graphs are always concave up or always concave down. Consequently, exponential functions do not have extrema except on a closed interval, and their graphs do not have inflection points.
- **2.3.A.4** If the values of the additive transformation function $g(x) = f(x) + k$ of any function f are proportional over equal-length input-value intervals, then f is exponential.
- **2.3.A.5** For an exponential function in general form, as the input values increase or decrease without bound, the output values will increase or decrease without bound or will get arbitrarily close to zero. That is, for an exponential function in general form, $\lim_{x \to \pm\infty} ab^x = \infty$, $\lim_{x \to \pm\infty} ab^x = -\infty$, or $\lim_{x \to \pm\infty} ab^x = 0$.

TOPIC 2.4

Exponential Function Manipulation

Skills Focus: 1.B, 3.A

LEARNING OBJECTIVES	ESSENTIAL KNOWLEDGE
2.4.A Rewrite exponential expressions in equivalent forms.	**2.4.A.1** The product property for exponents states that $b^m b^n = b^{(m+n)}$. Graphically, this property implies that every horizontal translation of an exponential function, $f(x) = b^{(x+k)}$, is equivalent to a vertical dilation, $f(x) = b^{(x+k)} = b^x b^k = ab^x$, where $a = b^k$.
	2.4.A.2 The power property for exponents states that $\left(b^m\right)^n = b^{(mn)}$. Graphically, this property implies that every horizontal dilation of an exponential function, $f(x) = b^{(cx)}$, is equivalent to a change of the base of an exponential function, $f(x) = \left(b^c\right)^x$, where b^c is a constant and $c \neq 0$.
	2.4.A.3 The negative exponent property states that $b^{-n} = \frac{1}{b^n}$.
	2.4.A.4 The value of an exponential expression involving an exponential unit fraction, such as $b^{(1/k)}$ where k is a natural number, is the $k\text{th}$ root of b, when it exists.

TOPIC 2.5

Exponential Function Context and Data Modeling

Skills Focus: 1.C, 3.B

LEARNING OBJECTIVES

2.5.A Construct a model for situations involving proportional output values over equal-length input-value intervals.

ESSENTIAL KNOWLEDGE

- **2.5.A.1** Exponential functions model growth patterns where successive output values over equal-length input-value intervals are proportional. When the input values are whole numbers, exponential functions model situations of repeated multiplication of a constant to an initial value.
- **2.5.A.2** A constant may need to be added to the dependent variable values of a data set to reveal a proportional growth pattern.
- **2.5.A.3** An exponential function model can be constructed from an appropriate ratio and initial value or from two input-output pairs. The initial value and the base can be found by solving a system of equations resulting from the two input-output pairs.
- **2.5.A.4** Exponential function models can be constructed by applying transformations to $f(x)=ab^x$ based on characteristics of a contextual scenario or data set.
- **2.5.A.5** Exponential function models can be constructed for a data set with technology using exponential regressions.
- **2.5.A.6** The natural base e, which is approximately 2.718, is often used as the base in exponential functions that model contextual scenarios.

LEARNING OBJECTIVES	ESSENTIAL KNOWLEDGE
2.5.B Apply exponential models to answer questions about a data set or contextual scenario.	▪ **2.5.B.1** For an exponential model in general form $f(x)=ab^x$, the base of the exponent, b, can be understood as a growth factor in successive unit changes in the input values and is related to a percent change in context. ▪ **2.5.B.2** Equivalent forms of an exponential function can reveal different properties of the function. For example, if d represents number of days, then the base of $f(d)=2^d$ indicates that the quantity increases by a factor of 2 every day, but the equivalent form $f(d)=\left(2^7\right)^{(d/7)}$ indicates that the quantity increases by a factor of 2^7 every week. ▪ **2.5.B.3** Exponential models can be used to predict values for the dependent variable, depending on the contextual constraints on the domain.

TOPIC 2.6

Competing Function Model Validation

Skills Focus: 2.A, 3.C

LEARNING OBJECTIVES	ESSENTIAL KNOWLEDGE
2.6.A Construct linear, quadratic, and exponential models based on a data set.	▪ **2.6.A.1** Two variables in a data set that demonstrate a slightly changing rate of change can be modeled by linear, exponential, and quadratic function models. ▪ **2.6.A.2** Models can be compared based on contextual clues and applicability to determine which model is most appropriate.
2.6.B Validate a model constructed from a data set.	▪ **2.6.B.1** A model is justified as *appropriate* for a data set if the graph of the residuals of a regression appear without pattern. ▪ **2.6.B.2** The difference between the predicted and actual values is the *error* in the model. Depending on the data set and context, it may be more appropriate to have an underestimate or overestimate for any given interval.

TOPIC 2.7

Composition of Functions

Skills Focus: 1.C, 2.B

LEARNING OBJECTIVES

2.7.A Evaluate the composition of two or more functions for given values.

ESSENTIAL KNOWLEDGE

- **2.7.A.1** If f and g are functions, the composite function $f\left(g\left(x\right)\right)$ maps a set of input values to a set of output values such that the output values of g are used as input values of f. For this reason, the domain of the composite function is restricted to those input values of g for which the corresponding output value is in the domain of f. The composite function $f\left(g\left(x\right)\right)$ uniquely maps input values of g to output values of f, dependent on the domain restrictions of f and g. The composite function $f\left(g\left(x\right)\right)$ can also be represented as $f \circ g\left(x\right)$.
- **2.7.A.2** Values for the composite function $f\left(g\left(x\right)\right)$ can be calculated or estimated from the analytical, graphical, numerical, or verbal representations of f and g by using output values from g as input values for f.
- **2.7.A.3** The composition of functions is not commutative; that is, $f\left(g\left(x\right)\right)$ and $g\left(f\left(x\right)\right)$ are typically different functions.
- **2.7.A.4** If the function $f\left(x\right)=x$ is composed with any function g, the resulting composite function is the same as g; that is, $g\left(f\left(x\right)\right)=f\left(g\left(x\right)\right)=g\left(x\right)$. The function $f\left(x\right)=x$ is called the *identity function*. When composing two functions, the identify function acts in the same way as 0, the additive identity, when adding two numbers and 1, the multiplicative identity, when multiplying two numbers.

LEARNING OBJECTIVES	ESSENTIAL KNOWLEDGE
2.7.B Construct a representation of the composition of two or more functions.	▪ **2.7.B.1** Function composition is useful for relating two quantities that are not directly related by an existing formula. ▪ **2.7.B.2** When analytic representations of the functions f and g are available, an analytic representation of $f\left(g\left(x\right)\right)$ can be constructed by substituting $g\left(x\right)$ for every instance of x in f. ▪ **2.7.B.3** A numerical or graphical representation of $f\left(g\left(x\right)\right)$ can often be constructed by calculating or estimating values for $\left(x, f\left(g\left(x\right)\right)\right)$.
2.7.C Rewrite a given function as a composite of two or more functions.	▪ **2.7.C.1** Functions given analytically can often be decomposed into less complex functions. When properly decomposed, the variable in one function should replace each instance of the function with which it was composed. ▪ **2.7.C.2** An additive transformation of a function, f, that results in vertical and horizontal translations can be understood as the composition of $g\left(x\right) = x + k$ with f. ▪ **2.7.C.3** A multiplicative transformation of a function, f, that results in vertical and horizontal dilations can be understood as the composition of $g\left(x\right) = kx$ with f.

TOPIC 2.8

Inverse Functions

Skills Focus: 1.A, 2.B

LEARNING OBJECTIVES	ESSENTIAL KNOWLEDGE
2.8.A Determine the input-output pairs of the inverse of a function.	▪ **2.8.A.1** On a specified domain, a function, f, has an inverse function, or is invertible, if each output value of f is mapped from a unique input value. The domain of a function may be restricted in many ways to make the function invertible. ▪ **2.8.A.2** An inverse function can be thought of as a reverse mapping of the function. An inverse function, f^{-1}, maps the output values of a function, f, on its invertible domain to their corresponding input values; that is, if $f(a)=b$, then $f^{-1}(b)=a$. Alternately, on its invertible domain, if a function consists of input-output pairs (a,b), then the inverse function consists of input-output pairs (b,a).
2.8.B Determine the inverse of a function on an invertible domain.	▪ **2.8.B.1** The composition of a function, f, and its inverse function, f^{-1}, is the identity function; that is, $f\left(f^{-1}(x)\right)=f^{-1}\left(f(x)\right)=x$. ▪ **2.8.B.2** On a function's invertible domain, the function's range and domain are the inverse function's domain and range, respectively. The inverse of the table of values of $y=f(x)$ can be found by reversing the input-output pairs; that is, (a,b) corresponds to (b,a). ▪ **2.8.B.3** The inverse of the graph of the function $y=f(x)$ can be found by reversing the roles of the *x*- and *y*-axes; that is, by reflecting the graph of the function over the graph of the identity function $h(x)=x$. ▪ **2.8.B.4** The inverse of the function can be found by determining the inverse operations to reverse the mapping. One method for finding the inverse of the function f is reversing the roles of x and y in the equation $y=f(x)$, then solving for $y=f^{-1}(x)$. ▪ **2.8.B.5** In addition to limiting the domain of a function to obtain an inverse function, contextual restrictions may also limit the applicability of an inverse function.

TOPIC 2.9

Logarithmic Expressions

Skills Focus: 1.B

LEARNING OBJECTIVES	ESSENTIAL KNOWLEDGE
2.9.A Evaluate logarithmic expressions.	▪ **2.9.A.1** The logarithmic expression $\log_b c$ is equal to, or represents, the value that the base b must be exponentially raised to in order to obtain the value c. That is, $\log_b c = a$ if and only if $b^a = c$, where a and c are constants, $b > 0$, and $b \neq 1$. (when the base of a logarithmic expression is not specified, it is understood as the common logarithm with a base of 10) ▪ **2.9.A.2** The values of some logarithmic expressions are readily accessible through basic arithmetic while other values can be estimated through the use of technology. ▪ **2.9.A.3** On a logarithmic scale, each unit represents a multiplicative change of the base of the logarithm. For example, on a standard scale, the units might be $0, 1, 2, \ldots$, while on a logarithmic scale, using log base 10, the units might be $10^0, 10^1, 10^2, \ldots$.

TOPIC 2.10

Inverses of Exponential Functions

Skills Focus: 1.C, 2.B

LEARNING OBJECTIVES

2.10.A Construct representations of the inverse of an exponential function with an initial value of 1.

ESSENTIAL KNOWLEDGE

- **2.10.A.1** The general form of a logarithmic function is $f(x) = a\log_b x$, with *base* b, where $b > 0$, $b \neq 1$, and $a \neq 0$.
- **2.10.A.2** The way in which input and output values vary together have an inverse relationship in exponential and logarithmic functions. Output values of general-form exponential functions change proportionately as input values increase in equal-length intervals. However, input values of general-form logarithmic functions change proportionately as output values increase in equal-length intervals. Alternately, exponential growth is characterized by output values changing multiplicatively as input values change additively, whereas logarithmic growth is characterized by output values changing additively as input values change multiplicatively.
- **2.10.A.3** $f(x) = \log_b x$ and $g(x) = b^x$, where $b > 0$ and $b \neq 1$, are inverse functions. That is, $g(f(x)) = f(g(x)) = x$.
- **2.10.A.4** The graph of the logarithmic function $f(x) = \log_b x$, where $b > 0$ and $b \neq 1$, is a reflection of the graph of the exponential function $g(x) = b^x$, where $b > 0$ and $b \neq 1$, over the graph of the identity function $h(x) = x$.
- **2.10.A.5** If (s, t) is an ordered pair of the exponential function $g(x) = b^x$, where $b > 0$ and $b \neq 1$, then (t, s) is an ordered pair of the logarithmic function $f(x) = \log_b x$, where $b > 0$ and $b \neq 1$.

TOPIC 2.11

Logarithmic Functions

Skills Focus: 3.A

LEARNING OBJECTIVES

2.11.A Identify key characteristics of logarithmic functions.

ESSENTIAL KNOWLEDGE

- **2.11.A.1** The domain of a logarithmic function in general form is any real number greater than zero, and its range is all real numbers.
- **2.11.A.2** Because logarithmic functions are inverses of exponential functions, logarithmic functions are also always increasing or always decreasing, and their graphs are either always concave up or always concave down. Consequently, logarithmic functions do not have extrema except on a closed interval, and their graphs do not have inflection points.
- **2.11.A.3** The additive transformation function $g(x)=f(x+k)$, where $k \neq 0$, of a logarithmic function f in general form does not have input values that are proportional over equal-length output-value intervals. However, if the output values of the additive transformation function, $g(x)=f(x+k)$,of any function f are proportional over equal-length input-value intervals, then f is logarithmic.
- **2.11.A.4** With their limited domain, logarithmic functions in general form are vertically asymptotic to $x=0$, with an end behavior that is unbounded. That is, for a logarithmic function in general form, $\lim_{x \to 0^+} a\log_b x = \pm\infty$ and $\lim_{x \to \infty} a\log_b x = \pm\infty$.

TOPIC 2.12

Logarithmic Function Manipulation

Skills Focus: 1.B, 3.A

LEARNING OBJECTIVES

2.12.A Rewrite logarithmic expressions in equivalent forms.

ESSENTIAL KNOWLEDGE

- **2.12.A.1** The product property for logarithms states that $\log_b(xy) = \log_b x + \log_b y$. Graphically, this property implies that every horizontal dilation of a logarithmic function, $f(x) = \log_b(kx)$, is equivalent to a vertical translation, $f(x) = \log_b(kx) = \log_b k + \log_b x = a + \log_b x$, where $a = \log_b k$.
- **2.12.A.2** The power property for logarithms states that $\log_b x^n = n\log_b x$. Graphically, this property implies that raising the input of a logarithmic function to a power, $f(x) = \log_b x^k$, results in a vertical dilation, $f(x) = \log_b x^k = k\log_b x$.
- **2.12.A.3** The change of base property for logarithms states that $\log_b x = \frac{\log_a x}{\log_a b}$, where $a > 0$ and $a \neq 1$. This implies that all logarithmic functions are vertical dilations of each other.
- **2.12.A.4** The function $f(x) = \ln x$ is a logarithmic function with the natural base e; that is, $\ln x = \log_e x$.

TOPIC 2.13

Exponential and Logarithmic Equations and Inequalities

Skills Focus: 1.A, 1.B, 1.C

LEARNING OBJECTIVES	ESSENTIAL KNOWLEDGE
2.13.A Solve exponential and logarithmic equations and inequalities.	▪ **2.13.A.1** Properties of exponents, properties of logarithms, and the inverse relationship between exponential and logarithmic functions can be used to solve equations and inequalities involving exponents and logarithms. ▪ **2.13.A.2** When solving exponential and logarithmic equations found through analytical or graphical methods, the results should be examined for extraneous solutions precluded by the mathematical or contextual limitations. ▪ **2.13.A.3** Logarithms can be used to rewrite expressions involving exponential functions in different ways that may reveal helpful information. Specifically, $b^x = c^{(\log_c b)(x)}$.
2.13.B Construct the inverse function for exponential and logarithmic functions.	▪ **2.13.B.1** The function $f(x) = ab^{(x-h)} + k$ is a combination of additive transformations of an exponential function in general form. The inverse of $y = f(x)$ can be found by determining the inverse operations to reverse the mapping. ▪ **2.13.B.2** The function $f(x) = a\log_b(x-h) + k$ is a combination of additive transformations of a logarithmic function in general form. The inverse of $y = f(x)$ can be found by determining the inverse operations to reverse the mapping.

TOPIC 2.14

Logarithmic Function Context and Data Modeling

Skills Focus: 1.C, 3.B

LEARNING OBJECTIVES

2.14.A Construct a logarithmic function model.

ESSENTIAL KNOWLEDGE

- **2.14.A.1** Logarithmic functions are inverses of exponential functions and can be used to model situations involving proportional growth, or repeated multiplication, where the input values change proportionally over equal-length output-value intervals. Alternately, if the output value is a whole number, it indicates how many times the initial value has been multiplied by the proportion.
- **2.14.A.2** A logarithmic function model can be constructed from an appropriate proportion and a real zero or from two input-output pairs.
- **2.14.A.3** Logarithmic function models can be constructed by applying transformations to $f(x) = a\log_b x$ based on characteristics of a context or data set.
- **2.14.A.4** Logarithmic function models can be constructed for a data set with technology using logarithmic regressions.
- **2.14.A.5** The natural log function is often useful in modeling natural phenomena.
- **2.14.A.6** Logarithmic function models can be used to predict values for the dependent variable.

TOPIC 2.15

Semi-log Plots

Skills Focus: 2.B, 3.C

LEARNING OBJECTIVES	ESSENTIAL KNOWLEDGE
2.15.A Determine if an exponential model is appropriate by examining a semi-log plot of a data set.	▪ **2.15.A.1** In a semi-log plot, one of the axes is logarithmically scaled. When the *y*-axis of a semi-log plot is logarithmically scaled, data or functions that demonstrate exponential characteristics will appear linear. ▪ **2.15.A.2** An advantage of semi-log plots is that a constant never needs to be added to the dependent variable values to reveal that an exponential model is appropriate.
2.15.B Construct the linearization of exponential data.	▪ **2.15.B.1** Techniques used to model linear functions can be applied to a semi-log graph. ▪ **2.15.B.2** For an exponential model of the form $y = ab^x$, the corresponding linear model for the semi-log plot is $y = (\log_n b)x + \log_n a$, where $n > 0$ and $n \neq 1$. Specifically, the linear rate of change is $\log_n b$, and the initial linear value is $\log_n a$.

UNIT 3:

Trigonometric and Polar Functions

Unit at a Glance

Topic #	Topic Title	Suggested Skill Focus
3.1	Periodic Phenomena	2.B, 3.A
3.2	Sine, Cosine, and Tangent	2.A, 3.A
3.3	Sine and Cosine Function Values	2.A, 3.B
3.4	Sine and Cosine Function Graphs	2.A, 3.A
3.5	Sinusoidal Functions	2.A, 3.A
3.6	Sinusoidal Function Transformations	1.C, 2.B
3.7	Sinusoidal Function Context and Data Modeling	1.C, 3.C
3.8	The Tangent Function	2.A, 3.A
3.9	Inverse Trigonometric Functions	1.C, 2.B
3.10	Trigonometric Equations and Inequalities	1.A, 2.A, 3.B
3.11	The Secant, Cosecant, and Cotangent Functions	2.B, 3.A
3.12	Equivalent Representations of Trigonometric Functions	1.A, 1.B, 3.B
3.13	Trigonometry and Polar Coordinates	1.B, 2.A
3.14	Polar Function Graphs	2.B, 3.A
3.15	Rates of Change in Polar Functions	3.A, 3.C

TOPIC 3.1

Periodic Phenomena

Skills Focus: 2.B, 3.A

LEARNING OBJECTIVES	ESSENTIAL KNOWLEDGE
3.1.A Construct graphs of periodic relationships based on verbal representations.	▪ **3.1.A.1** A periodic relationship can be identified between two aspects of a context if, as the input values increase, the output values demonstrate a repeating pattern over successive equal-length intervals. ▪ **3.1.A.2** The graph of a periodic relationship can be constructed from the graph of a single cycle of the relationship.
3.1.B Describe key characteristics of a periodic function based on a verbal representation.	▪ **3.1.B.1** The *period* of the function is the smallest positive value k such that $f(x+k)=f(x)$ for all x in the domain. Consequently, the behavior of a periodic function is completely determined by any interval of width k. ▪ **3.1.B.2** The period can be estimated by investigating successive equal-length output values and finding where the pattern begins to repeat. ▪ **3.1.B.3** Periodic functions take on characteristics of other functions, such as intervals of increase and decrease, different concavities, and various rates of change. However, with periodic functions, all characteristics found in one period of the function will be in every period of the function.

TOPIC 3.2

Sine, Cosine, and Tangent

Skills Focus: 2.A, 3.A

LEARNING OBJECTIVES

3.2.A Determine the sine, cosine, and tangent of an angle using the unit circle.

ESSENTIAL KNOWLEDGE

- **3.2.A.1** In the coordinate plane, an angle is in *standard position* when the vertex coincides with the origin and one ray coincides with the positive *x*-axis. The other ray is called the *terminal ray*. Positive and negative angle measures indicate rotations from the positive *x*-axis in the counterclockwise and clockwise direction, respectively. Angles in standard position that share a terminal ray differ by an integer number of revolutions.
- **3.2.A.2** The radian measure of an angle in standard position is the ratio of the length of the arc of a circle centered at the origin subtended by the angle to the radius of that same circle. For a unit circle, which has radius 1, the radian measure is the same as the length of the subtended arc.
- **3.2.A.3** Given an angle in standard position and a circle centered at the origin, there is a point, P, where the terminal ray intersects the circle. The *sine* of the angle is the ratio of the vertical displacement of P from the *x*-axis to the distance between the origin and point P. Therefore, for a unit circle, the sine of the angle is the *y*-coordinate of point P.
- **3.2.A.4** Given an angle in standard position and a circle centered at the origin, there is a point, P, where the terminal ray intersects the circle. The *cosine* of the angle is the ratio of the horizontal displacement of P from the *y*-axis to the distance between the origin and point P. Therefore, for a unit circle, the cosine of the angle is the *x*-coordinate of point P.
- **3.2.A.5** Given an angle in standard position, the *tangent* of the angle is the slope, if it exists, of the terminal ray. Because the slope of the terminal ray is the ratio of the vertical displacement to the horizontal displacement over any interval, the tangent of the angle is the ratio of the *y*-coordinate to the *x*-coordinate of the point at which the terminal ray intersects the unit circle; alternately, it is the ratio of the angle's sine to its cosine.

TOPIC 3.3

Sine and Cosine Function Values

Skills Focus: 2.A, 3.B

LEARNING OBJECTIVES	ESSENTIAL KNOWLEDGE
3.3.A Determine coordinates of points on a circle centered at the origin.	▪ **3.3.A.1** Given an angle of measure θ in standard position and a circle with radius r centered at the origin, there is a point, P, where the terminal ray intersects the circle. The coordinates of point P are $\left(r\cos\theta, r\sin\theta\right)$. ▪ **3.3.A.2** The geometry of isosceles right and equilateral triangles, while attending to the signs of the values based on the quadrant of the angle, can be used to find exact values for the cosine and sine of angles that are multiples of $\frac{\pi}{4}$ and $\frac{\pi}{6}$ radians and whose terminal rays do not lie on an axis.

TOPIC 3.4

Sine and Cosine Function Graphs

Skills Focus: 2.A, 3.A

LEARNING OBJECTIVES

3.4.A Construct representations of the sine and cosine functions using the unit circle.

ESSENTIAL KNOWLEDGE

- **3.4.A.1** Given an angle of measure θ in standard position and a unit circle centered at the origin, there is a point, P, where the terminal ray intersects the circle. The sine function, $f(\theta)=\sin\theta$, gives the *y*-coordinate, or vertical displacement from the *x*-axis, of point P. The domain of the sine function is all real numbers.
- **3.4.A.2** As the input values, or angles, of the sine function increase, the output values oscillate between negative one and one, taking every value in between and tracking the vertical distance of points on the unit circle from the *x*-axis.
- **3.4.A.3** Given an angle of measure θ in standard position and a unit circle centered at the origin, there is a point, P, where the terminal ray intersects the circle. The cosine function, $f(\theta)=\cos\theta$, gives the *x*-coordinate, or horizontal displacement from the *y*-axis, of point P. The domain of the cosine function is all real numbers.
- **3.4.A.4** As the input values, or angles, of the cosine function increase, the output values oscillate between negative one and one, taking every value in between and tracking the horizontal distance of points on the unit circle from the *y*-axis.

TOPIC 3.5

Sinusoidal Functions

Skills Focus: 2.A, 3.A

LEARNING OBJECTIVES

3.5.A Identify key characteristics of the sine and cosine functions.

ESSENTIAL KNOWLEDGE

- **3.5.A.1** A *sinusoidal function* is any function that involves additive and multiplicative transformations of $f(\theta) = \sin\theta$. The sine and cosine functions are both sinusoidal functions, with $\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right)$.
- **3.5.A.2** The period and frequency of a sinusoidal function are reciprocals. The period of $f(\theta) = \sin\theta$ and $g(\theta) = \cos\theta$ is 2π, and the frequency is $\frac{1}{2\pi}$.
- **3.5.A.3** The amplitude of a sinusoidal function is half the difference between its maximum and minimum values. The amplitude of $f(\theta) = \sin\theta$ and $g(\theta) = \cos\theta$ is 1.
- **3.5.A.4** The midline of the graph of a sinusoidal function is determined by the average, or arithmetic mean, of the maximum and minimum values of the function. The midline of the graphs of $y = \sin\theta$ and $y = \cos\theta$ is $y = 0$.
- **3.5.A.5** As input values increase, the graphs of sinusoidal functions oscillate between concave down and concave up.
- **3.5.A.6** The graph of $y = \sin\theta$ has rotational symmetry about the origin and is therefore an odd function. The graph of $y = \cos\theta$ has reflective symmetry over the *y*-axis and is therefore an even function.

TOPIC 3.6

Sinusoidal Function Transformations

Skills Focus: 1.C, 2.B

LEARNING OBJECTIVES

3.6.A Identify the amplitude, vertical shift, period, and phase shift of a sinusoidal function.

ESSENTIAL KNOWLEDGE

- **3.6.A.1** Functions that can be written in the form $f(\theta)=a\sin(b(\theta+c))+d$ or $g(\theta)=a\cos(b(\theta+c))+d$, where a, b, c, and d are real numbers and $a \neq 0$, are sinusoidal functions and are transformations of the sine and cosine functions. Additive and multiplicative transformations are the same for both sine and cosine, because the cosine function is a phase shift of the sine function by $-\frac{\pi}{2}$ units.
- **3.6.A.2** The graph of the additive transformation $g(\theta)=\sin\theta+d$ of the sine function $f(\theta)=\sin\theta$ is a vertical translation of the graph of f, including its midline, by d units. The same transformation of the cosine function yields the same result.
- **3.6.A.3** The graph of the additive transformation $g(\theta)=\sin(\theta+c)$ of the sine function $f(\theta)=\sin\theta$ is a horizontal translation, or phase shift, of the graph of f by $-c$ units. The same transformation of the cosine function yields the same result.
- **3.6.A.4** The graph of the multiplicative transformation $g(\theta)=a\sin\theta$ of the sine function $f(\theta)=\sin\theta$ is a vertical dilation of the graph of f and differs in amplitude by a factor of $|a|$. The same transformation of the cosine function yields the same result.
- **3.6.A.5** The graph of the multiplicative transformation $g(\theta)=\sin(b\theta)$ of the sine function $f(\theta)=\sin\theta$ is a horizontal dilation of the graph of f and differs in period by a factor of $\left|\frac{1}{b}\right|$. The same transformation of the cosine function yields the same result.

LEARNING OBJECTIVES	ESSENTIAL KNOWLEDGE
3.6.A Identify the amplitude, vertical shift, period, and phase shift of a sinusoidal function.	▪ **3.6.A.6** The graph of $y = f(\theta) = a\sin(b(\theta + c)) + d$ has an amplitude of $\lvert a \rvert$ units, a period of $\left\lvert \frac{1}{b} \right\rvert 2\pi$ units, a midline vertical shift of d units from $y = 0$, and a phase shift of $-c$ units. The same transformations of the cosine function yield the same results.

TOPIC 3.7

Sinusoidal Function Context and Data Modeling

Skills Focus: 1.C, 3.C

LEARNING OBJECTIVES	ESSENTIAL KNOWLEDGE
3.7.A Construct sinusoidal function models of periodic phenomena by estimating key values.	▪ **3.7.A.1** The smallest interval of input values over which the maximum or minimum output values start to repeat can be used to determine or estimate the period and frequency for a sinusoidal function model. ▪ **3.7.A.2** The maximum and minimum output values can be used to determine or estimate the amplitude and vertical shift for a sinusoidal function model. ▪ **3.7.A.3** An actual pair of input-output values can be compared to pairs of input-output values produced by a sinusoidal function model to determine or estimate a phase shift for the model. ▪ **3.7.A.4** Technology can be used to find an appropriate sinusoidal function model for a data set. ▪ **3.7.A.5** Sinusoidal functions that model a data set are frequently only useful over their contextual domain and can be used to predict values of the dependent variable from a value of the independent variable.

TOPIC 3.8

The Tangent Function

Skills Focus: 2.A, 3.A

LEARNING OBJECTIVES

3.8.A Construct representations of the tangent function using the unit circle.

ESSENTIAL KNOWLEDGE

- **3.8.A.1** Given an angle of measure θ in standard position and a unit circle centered at the origin, there is a point, P, where the terminal ray intersects the circle. The tangent function, $f(\theta) = \tan\theta$, gives the slope of the terminal ray.
- **3.8.A.2** Because the slope of the terminal ray is the ratio of the change in the *y*-values to the change in the *x*-values between any two points on the ray, the tangent function is also the ratio of the sine function to the cosine function. Therefore, $\tan\theta = \frac{\sin\theta}{\cos\theta}$, where $\cos\theta \neq 0$.

LEARNING OBJECTIVES

3.8.B Describe key characteristics of the tangent function.

ESSENTIAL KNOWLEDGE

- **3.8.B.1** Because the slope values of the terminal ray repeat every one-half revolution of the circle, the tangent function has a period of π.
- **3.8.B.2** The tangent function demonstrates periodic asymptotic behavior at input values $\theta = \frac{\pi}{2} + k\pi$, for integer values of k, because $\cos\theta = 0$ at those values.
- **3.8.B.3** The tangent function increases and its graph changes from concave down to concave up between consecutive asymptotes.

LEARNING OBJECTIVES

3.8.C Describe additive and multiplicative transformations involving the tangent function.

ESSENTIAL KNOWLEDGE

- **3.8.C.1** The graph of the additive transformation $g(\theta) = \tan\theta + d$ of the tangent function $f(\theta) = \tan\theta$ is a vertical translation of the graph of f and the line containing its inflection points by d units.
- **3.8.C.2** The graph of the additive transformation $g(\theta) = \tan(\theta + c)$ of the tangent function $f(\theta) = \tan\theta$ is a horizontal translation, or phase shift, of the graph of f by $-c$ units.
- **3.8.C.3** The graph of the multiplicative transformation $g(\theta) = a\tan\theta$ of the tangent function $f(\theta) = \tan\theta$ is a vertical dilation of the graph of f by a factor of $|a|$. If $a < 0$, the transformation involves a reflection over the *x*-axis.
- **3.8.C.4** The graph of the multiplicative transformation $g(\theta) = \tan(b\theta)$ of the tangent function $f(\theta) = \tan\theta$ is a horizontal dilation of the graph of f and differs in period by a factor of $\left|\frac{1}{b}\right|$. If $b < 0$, the transformation involves a reflection over the *y*-axis.
- **3.8.C.5** The graph of $y = f(\theta) = a\tan(b(\theta + c)) + d$ is a vertical dilation of the graph of $y = \tan\theta$ by a factor of $|a|$, has a period of $\left|\frac{1}{b}\right|\pi$ units, is a vertical shift of the line containing the inflection points of the graph of $y = \tan\theta$ by d units, and is a phase shift of $-c$ units.

TOPIC 3.9

Inverse Trigonometric Functions

Skills Focus: 1.C, 2.B

LEARNING OBJECTIVES

3.9.A Construct analytical and graphical representations of the inverse of the sine, cosine, and tangent functions over a restricted domain.

ESSENTIAL KNOWLEDGE

- **3.9.A.1** For inverse trigonometric functions, the input and output values are switched from their corresponding trigonometric functions, so the output value of an inverse trigonometric function is often interpreted as an angle measure and the input is a value in the range of the corresponding trigonometric function.
- **3.9.A.2** The inverse trigonometric functions are called *arcsine*, *arccosine*, and *arctangent* (also represented as $\sin^{-1} x$, $\cos^{-1} x$, and $\tan^{-1} x$) . Because the corresponding trigonometric functions are periodic, they are only invertible if they have restricted domains.
- **3.9.A.3** In order to define their respective inverse functions, the domain of the sine function is restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the cosine function to $\left[0, \pi\right]$, and the tangent function to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

TOPIC 3.10

Trigonometric Equations and Inequalities

Skills Focus: 1.A, 2.A, 3.B

LEARNING OBJECTIVES	ESSENTIAL KNOWLEDGE
3.10.A Solve equations and inequalities involving trigonometric functions.	▪ **3.10.A.1** Inverse trigonometric functions are useful in solving equations and inequalities involving trigonometric functions, but solutions may need to be modified due to domain restrictions. ▪ **3.10.A.2** Because trigonometric functions are periodic, there are often infinite solutions to trigonometric equations. ▪ **3.10.A.3** In trigonometric equations and inequalities arising from a contextual scenario, there is often a domain restriction that can be implied from the context, which limits the number of solutions.

TOPIC 3.11

The Secant, Cosecant, and Cotangent Functions

Skills Focus: 2.B, 3.A

LEARNING OBJECTIVES

3.11.A Identify key characteristics of functions that involve quotients of the sine and cosine functions.

ESSENTIAL KNOWLEDGE

- **3.11.A.1** The secant function, $f(\theta) = \sec\theta$, is the reciprocal of the cosine function, where $\cos\theta \neq 0$.
- **3.11.A.2** The cosecant function, $f(\theta) = \csc\theta$, is the reciprocal of the sine function, where $\sin\theta \neq 0$.
- **3.11.A.3** The graphs of the secant and cosecant functions have vertical asymptotes where cosine and sine are zero, respectively, and have a range of $(-\infty, -1] \cup [1, \infty)$.
- **3.11.A.4** The cotangent function, $f(\theta) = \cot\theta$, is the reciprocal of the tangent function, where $\tan\theta \neq 0$. Equivalently, $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$, where $\sin\theta \neq 0$.
- **3.11.A.5** The graph of the cotangent function has vertical asymptotes for domain values where $\tan\theta = 0$ and is decreasing between consecutive asymptotes.

TOPIC 3.12

Equivalent Representations of Trigonometric Functions

Skills Focus: 1.A, 1.B, 3.B

LEARNING OBJECTIVES

3.12.A Rewrite trigonometric expressions in equivalent forms with the Pythagorean identity.

ESSENTIAL KNOWLEDGE

- **3.12.A.1** The Pythagorean theorem can be applied to right triangles with points on the unit circle at coordinates $(\cos\theta, \sin\theta)$, resulting in the Pythagorean identity: $\sin^2\theta + \cos^2\theta = 1$.
- **3.12.A.2** The Pythagorean identity can be algebraically manipulated into other forms involving trigonometric functions, such as $\tan^2\theta = \sec^2\theta - 1$, and can be used to establish other trigonometric relationships, such as $\arcsin x = \arccos\sqrt{1-x^2}$, with appropriate domain restrictions.

LEARNING OBJECTIVES

3.12.B Rewrite trigonometric expressions in equivalent forms with sine and cosine sum identities.

ESSENTIAL KNOWLEDGE

- **3.12.B.1** The sum identity for sine is $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.
- **3.12.B.2** The sum identity for cosine is $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.
- **3.12.B.3** The sum identities for sine and cosine can also be used as difference and double-angle identities.
- **3.12.B.4** Properties of trigonometric functions, known trigonometric identities, and other algebraic properties can be used to verify additional trigonometric identities.

LEARNING OBJECTIVES

3.12.C Solve equations using equivalent analytic representations of trigonometric functions.

ESSENTIAL KNOWLEDGE

- **3.12.C.1** A specific equivalent form involving trigonometric expressions can make information more accessible.
- **3.12.C.2** Equivalent trigonometric forms may be useful in solving trigonometric equations and inequalities.

TOPIC 3.13

Trigonometry and Polar Coordinates

Skills Focus: 1.B, 2.A

LEARNING OBJECTIVES

3.13.A Determine the location of a point in the plane using both rectangular and polar coordinates.

ESSENTIAL KNOWLEDGE

- **3.13.A.1** The polar coordinate system is based on a grid of circles centered at the origin and on lines through the origin. Polar coordinates are defined as an ordered pair, (r, θ), such that $|r|$ represents the radius of the circle on which the point lies, and θ represents the measure of an angle in standard position whose terminal ray includes the point. In the polar coordinate system, the same point can be represented many ways.
- **3.13.A.2** The coordinates of a point in the polar coordinate system, (r, θ), can be converted to coordinates in the rectangular coordinate system, (x, y), using $x = r\cos\theta$ and $y = r\sin\theta$.
- **3.13.A.3** The coordinates of a point in the rectangular coordinate system, (x, y), can be converted to coordinates in the polar coordinate system, (r, θ), using $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan\frac{y}{x}$ for $x > 0$ or $\theta = \arctan\frac{y}{x} + \pi$ for $x < 0$.
- **3.13.A.4** A complex number can be understood as a point in the complex plane and can be determined by its corresponding rectangular or polar coordinates. When the complex number has the rectangular coordinates (a, b), it can be expressed as $a + bi$. When the complex number has polar coordinates (r, θ), it can be expressed as $(r\cos\theta) + i(r\sin\theta)$.

TOPIC 3.14

Polar Function Graphs

Skills Focus: 2.B, 3.A

LEARNING OBJECTIVES

3.14.A Construct graphs of polar functions.

ESSENTIAL KNOWLEDGE

- **3.14.A.1** The graph of the function $r = f(\theta)$ in polar coordinates consists of input-output pairs of values where the input values are angle measures and the output values are radii.
- **3.14.A.2** The domain of the polar function $r = f(\theta)$, given graphically, can be restricted to a desired portion of the function by selecting endpoints corresponding to the desired angle and radius.
- **3.14.A.3** When graphing polar functions in the form of $r = f(\theta)$, changes in input values correspond to changes in angle measure from the positive *x*-axis, and changes in output values correspond to changes in distance from the origin.

TOPIC 3.15

Rates of Change in Polar Functions

Skills Focus: 3.A, 3.C

LEARNING OBJECTIVES

3.15.A Describe characteristics of the graph of a polar function.

ESSENTIAL KNOWLEDGE

- **3.15.A.1** If a polar function, $r = f(\theta)$, is positive and increasing or negative and decreasing, then the distance between $f(\theta)$ and the origin is increasing.
- **3.15.A.2** If a polar function, $r = f(\theta)$, is positive and decreasing or negative and increasing, then the distance between $f(\theta)$ and the origin is decreasing.
- **3.15.A.3** For a polar function, $r = f(\theta)$, if the function changes from increasing to decreasing or decreasing to increasing on an interval, then the function has a relative extremum on the interval corresponding to a point relatively closest to or farthest from the origin.
- **3.15.A.4** The average rate of change of r with respect to θ over an interval of θ is the ratio of the change in the radius values to the change in θ over an interval of θ. Graphically, the average rate of change indicates the rate at which the radius is changing per radian.
- **3.15.A.5** The average rate of change of r with respect to θ over an interval of θ can be used to estimate values of the function within the interval.

UNIT 4:

Functions Involving Parameters, Vectors, and Matrices

Unit 4 will **not** be assessed on the AP Precalculus Exam.
It's provided in the course framework as a courtesy for teachers who'd like to include these topics.

Unit at a Glance

Topic #	Topic Title	Suggested Skill Focus
4.1	Parametric Functions	1.A, 2.B
4.2	Parametric Functions Modeling Planar Motion	3.A, 3.B
4.3	Parametric Functions and Rates of Change	3.B, 3.C
4.4	Parametrically Defined Circles and Lines	1.B, 1.C
4.5	Implicitly Defined Functions	2.B, 3.A
4.6	Conic Sections	1.B, 2.A, 2.B
4.7	Parametrization of Implicitly Defined Functions	1.B, 2.A
4.8	Vectors	2.A, 3.A, 3.B
4.9	Vector-Valued Functions	3.C
4.10	Matrices	1.B, 3.B
4.11	The Inverse and Determinant of a Matrix	1.B, 3.B
4.12	Linear Transformations and Matrices	1.B
4.13	Matrices as Functions	1.B, 2.A, 3.A
4.14	Matrices Modeling Contexts	1.C, 3.B, 3.C

TOPIC 4.1

Parametric Functions

Skills Focus: 1.A, 2.B

LEARNING OBJECTIVES

4.1.A Construct a graph or table of values for a parametric function represented analytically.

ESSENTIAL KNOWLEDGE

- **4.1.A.1** A parametric function in R^2, the set of all ordered pairs of two real numbers, consists of a set of two parametric equations in which two dependent variables, x and y, are dependent on a single independent variable, t, called the *parameter*.
- **4.1.A.2** Because variables x and y are dependent on the independent variable, t, the coordinates (x_i, y_i) at time t_i can be written as functions of t and can be expressed as the single parametric function $f(t)=(x(t), y(t))$, where in this case x and y are names of two functions.
- **4.1.A.3** A numerical table of values can be generated for the parametric function $f(t)=(x(t), y(t))$ by evaluating x_i and y_i at several values of t_i within the domain.
- **4.1.A.4** A graph of a parametric function can be sketched by connecting several points from the numerical table of values in order of increasing value of t.
- **4.1.A.5** The domain of the parametric function f is often restricted, which results in start and end points on the graph of f.

TOPIC 4.2

Parametric Functions Modeling Planar Motion

Skills Focus: 3.A, 3.B

LEARNING OBJECTIVES	ESSENTIAL KNOWLEDGE
4.2.A Identify key characteristics of a parametric planar motion function that are related to position.	▪ **4.2.A.1** A parametric function given by $f(t)=(x(t), y(t))$ can be used to model particle motion in the plane. The graph of this function indicates the position of a particle at time t. ▪ **4.2.A.2** The horizontal and vertical extrema of a particle's motion can be determined by identifying the maximum and minimum values of the functions $x(t)$ and $y(t)$, respectively. ▪ **4.2.A.3** The real zeros of the function $x(t)$ correspond to *y*-intercepts, and the real zeros of $y(t)$ correspond to *x*-intercepts.

TOPIC 4.3

Parametric Functions and Rates of Change

Skills Focus: 3.B, 3.C

LEARNING OBJECTIVES

4.3.A Identify key characteristics of a parametric planar motion function that are related to direction and rate of change.

ESSENTIAL KNOWLEDGE

- **4.3.A.1** As the parameter increases, the direction of planar motion of a particle can be analyzed in terms of x and y independently. If $x(t)$ is increasing or decreasing, the direction of motion is to the right or left, respectively. If $y(t)$ is increasing or decreasing, the direction of motion is up or down, respectively.
- **4.3.A.2** At any given point in the plane, the direction of planar motion may be different for different values of t.
- **4.3.A.3** The same curve in the plane can be parametrized in different ways and can be traversed in different directions with different parametric functions.
- **4.3.A.4** Over a given interval $[t_1, t_2]$ within the domain, the average rate of change can be computed for $x(t)$ and $y(t)$ independently. The ratio of the average rate of change of y to the average rate of change of x gives the slope of the graph between the points on the curve corresponding to t_1 and t_2, so long as the average rate of change of $x(t) \neq 0$.

TOPIC 4.4

Parametrically Defined Circles and Lines

Skills Focus: 1.B, 1.C

LEARNING OBJECTIVES

4.4.A Express motion around a circle or along a line segment parametrically.

ESSENTIAL KNOWLEDGE

- **4.4.A.1** A complete counterclockwise revolution around the unit circle that starts and ends at $(1,0)$ and is centered at the origin can be modeled by $(x(t), y(t)) = (\cos t, \sin t)$ with domain $0 \le t \le 2\pi$.
- **4.4.A.2** Transformations of the parametric function $(x(t), y(t)) = (\cos t, \sin t)$ can model any circular path traversed in the plane.
- **4.4.A.3** A linear path along the line segment from the point (x_1, y_1) to the point (x_2, y_2) can be parametrized many ways, including using an initial position (x_1, y_1) and rates of change for x with respect to t and y with respect to t.

TOPIC 4.5

Implicitly Defined Functions

Skills Focus: 2.B, 3.A

LEARNING OBJECTIVES	ESSENTIAL KNOWLEDGE
4.5.A Construct a graph of an equation involving two variables.	- **4.5.A.1** An equation involving two variables can implicitly describe one or more functions. - **4.5.A.2** An equation involving two variables can be graphed by finding solutions to the equation. - **4.5.A.3** Solving for one of the variables in an equation involving two variables can define a function whose graph is part or all of the graph of the equation.
4.5.B Determine how the two quantities related in an implicitly defined function vary together.	- **4.5.B.1** For ordered pairs on the graph of an implicitly defined function that are close together, if the ratio of the change in the two variables is positive, then the two variables simultaneously increase or both decrease; conversely, if the ratio is negative, then as one variable increases, the other decreases. - **4.5.B.2** The rate of change of x with respect to y or of y with respect to x can be zero, indicating vertical or horizontal intervals, respectively.

TOPIC 4.6

Conic Sections

Skills Focus: 1.B, 2.A, 2.B

LEARNING OBJECTIVES

4.6.A Represent conic sections with horizontal or vertical symmetry analytically.

ESSENTIAL KNOWLEDGE

- **4.6.A.1** A parabola with vertex (h, k) can, if $a \neq 0$, be represented analytically as $(y-k)^2 = a(x-h)$ if it opens left or right, or as $a(y-k) = (x-h)^2$ if it opens up or down.
- **4.6.A.2** An ellipse centered at (h, k) with horizontal radius a and vertical radius b can be represented analytically as $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. A circle is a special case of an ellipse where $a = b$.
- **4.6.A.3** A hyperbola centered at (h, k) with vertical and horizonal lines of symmetry can be represented algebraically as $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ for a hyperbola opening left and right, or as $-\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ for a hyperbola opening up and down.

TOPIC 4.7

Parametrization of Implicitly Defined Functions

Skills Focus: 1.B, 2.A

LEARNING OBJECTIVES	ESSENTIAL KNOWLEDGE
4.7.A Represent a curve in the plane parametrically.	**4.7.A.1** A parametrization $\left(x(t), y(t)\right)$ for an implicitly defined function will, when $x(t)$ and $y(t)$ are substituted for x and y, respectively, satisfy the corresponding equation for every value of t in the domain. **4.7.A.2** If f is a function of x, then $y=f(x)$ can be parametrized as $\left(x(t), y(t)\right)=\left(t, f(t)\right)$. If f is invertible, its inverse can be parametrized as $\left(x(t), y(t)\right)=\left(f(t), t\right)$ for an appropriate interval of t.
4.7.B Represent conic sections parametrically.	**4.7.B.1** A parabola can be parametrized in the same way that any equation that can be solved for x or y can be parametrized. Equations that can be solved for x can be parametrized as $\left(x(t), y(t)\right)=\left(f(t), t\right)$ by solving for x and replacing y with t. Equations that can be solved for y can be parametrized as $\left(x(t), y(t)\right)=\left(t, f(t)\right)$ by solving for y and replacing x with t. **4.7.B.2** An ellipse can be parametrized using the trigonometric functions $x(t)=h+a\cos t$ and $y(t)=k+b\sin t$ for $0\le t\le 2\pi$. **4.7.B.3** A hyperbola can be parametrized using trigonometric functions. For a hyperbola that opens left and right, the functions are $x(t)=h+a\sec t$ and $y(t)=k+b\tan t$ for $0\le t\le 2\pi$. For a hyperbola that opens up and down, the functions are $x(t)=h+a\tan t$ and $y(t)=k+b\sec t$ for $0\le t\le 2\pi$.

TOPIC 4.8

Vectors

Skills Focus: 2.A, 3.A, 3.B

LEARNING OBJECTIVES

4.8.A Identify characteristics of a vector.

ESSENTIAL KNOWLEDGE

- **4.8.A.1** A vector is a directed line segment. When a vector is placed in the plane, the point at the beginning of the line segment is called the *tail*, and the point at the end of the line segment is called the *head*. The length of the line segment is the *magnitude* of the vector.
- **4.8.A.2** A vector $\overrightarrow{P_1P_2}$ with two components can be plotted in the *xy*-plane from $P_1=(x_1, y_1)$ to $P_2=(x_2, y_2)$. The vector is identified by a and b, where $a=x_2-x_1$ and $b=y_2-y_1$. The vector can be expressed as $\langle a, b\rangle$. A zero vector $\langle 0, 0\rangle$ is the trivial case when $P_1=P_2$.
- **4.8.A.3** The *direction* of the vector is parallel to the line segment from the origin to the point with coordinates (a, b). The magnitude of the vector is the square root of the sum of the squares of the components.
- **4.8.A.4** For a vector represented geometrically in the plane, the components of the vector can be found using trigonometry.

LEARNING OBJECTIVES

4.8.B Determine sums and products involving vectors.

ESSENTIAL KNOWLEDGE

- **4.8.B.1** The multiplication of a constant and a vector results in a new vector whose components are found by multiplying the constant by each of the components of the original vector. The new vector is parallel to the original vector.
- **4.8.B.2** The sum of two vectors in R^2 is a new vector whose components are found by adding the corresponding components of the original vectors. The new vector can be represented graphically as a vector whose tail corresponds to the tail of the first vector and whose head corresponds to the head of the second vector when the second vector's tail is located at the first vector's head.
- **4.8.B.3** The dot product of two vectors is the sum of the products of their corresponding components. That is, $\langle a_1, b_1\rangle \cdot \langle a_2, b_2\rangle = a_1a_2+b_1b_2$.

LEARNING OBJECTIVES	ESSENTIAL KNOWLEDGE
4.8.C Determine a unit vector for a given vector.	▪ **4.8.C.1** A *unit vector* is a vector of magnitude 1. A unit vector in the same direction as a given nonzero vector can be found by scalar multiplying the vector by the reciprocal of its magnitude. ▪ **4.8.C.2** The vector $\langle a, b\rangle$ can be expressed as $a\vec{i} + b\vec{j}$ in R^2, where $\vec{i}$ and $\vec{j}$ are unit vectors in the x and y directions, respectively. That is, $\vec{i} = \langle 1, 0\rangle$ and $\vec{j} = \langle 0, 1\rangle$.
4.8.D Determine angles between vectors and magnitudes of vectors involved in vector addition.	▪ **4.8.D.1** The dot product is geometrically equivalent to the product of the magnitudes of the two vectors and the cosine of the angle between them. Therefore, if the dot product of two nonzero vectors is zero, then the vectors are perpendicular. ▪ **4.8.D.2** The Law of Sines and Law of Cosines can be used to determine side lengths and angles of triangles formed by vector addition.

TOPIC 4.9

Vector-Valued Functions

Skills Focus: 3.C

LEARNING OBJECTIVES

4.9.A Represent planar motion in terms of vector-valued functions.

ESSENTIAL KNOWLEDGE

- **4.9.A.1** The position of a particle moving in a plane that is given by the parametric function $f(t)=(x(t), y(t))$ may be expressed as a *vector-valued function*, $p(t)=x(t)\vec{i}+y(t)\vec{j}$ or $p(t)=\langle x(t), y(t)\rangle$. The magnitude of the position vector at time t gives the distance of the particle from the origin.
- **4.9.A.2** The vector-valued function $v(t)=\langle x(t), y(t)\rangle$ can be used to express the velocity of a particle moving in a plane at different times, t. At time t, the sign of $x(t)$ indicates if the particle is moving right or left, and the sign of $y(t)$ indicates if the particle is moving up or down. The magnitude of the velocity vector at time t gives the speed of the particle.

TOPIC 4.10

Matrices

Skills Focus: 1.B, 3.B

LEARNING OBJECTIVES	ESSENTIAL KNOWLEDGE
4.10.A Determine the product of two matrices.	▪ **4.10.A.1** An $n \times m$ matrix is an array consisting of n rows and m columns. ▪ **4.10.A.2** Two matrices can be multiplied if the number of columns in the first matrix equals the number of rows in the second matrix. The product of the matrices is a new matrix in which the component in the $i\text{th}$ row and $j\text{th}$ column is the dot product of the $i\text{th}$ row of the first matrix and the $j\text{th}$ column of the second matrix.

TOPIC 4.11

The Inverse and Determinant of a Matrix

Skills Focus: 1.B, 3.B

LEARNING OBJECTIVES

4.11.A Determine the inverse of a 2×2 matrix.

4.11.B Apply the value of the determinant to invertibility and vectors.

ESSENTIAL KNOWLEDGE

- **4.11.A.1** The identity matrix, I, is a square matrix consisting of ones on the diagonal from the top left to bottom right and zeros everywhere else.
- **4.11.A.2** Multiplying a square matrix by its corresponding identity matrix results in the original square matrix.
- **4.11.A.3** The product of a square matrix and its inverse, when it exists, is the identity matrix of the same size.
- **4.11.A.4** The inverse of a 2×2 matrix, when it exists, can be calculated with or without technology.

- **4.11.B.1** The determinant of the matrix $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad-bc$. The determinant can be calculated with or without technology and is denoted $\det(A)$.
- **4.11.B.2** If a 2×2 matrix consists of two column or row vectors from R^2, then the nonzero absolute value of the determinant of the matrix is the area of the parallelogram spanned by the vectors represented in the columns or rows of the matrix. If the determinant equals zero, then the vectors are parallel.
- **4.11.B.3** The square matrix A has an inverse if and only if $\det(A)\neq0$.

TOPIC 4.12

Linear Transformations and Matrices

Skills Focus: 1.B

LEARNING OBJECTIVES

4.12.A Determine the output vectors of a linear transformation using a 2×2 matrix.

ESSENTIAL KNOWLEDGE

- **4.12.A.1** A *linear transformation* is a function that maps an input vector to an output vector such that each component of the output vector is the sum of constant multiples of the input vector components.
- **4.12.A.2** A linear transformation will map the zero vector to the zero vector.
- **4.12.A.3** A single vector in R^2 can be expressed as a 2×1 matrix. A set of n vectors in R^2 can be expressed as a $2 \times n$ matrix.
- **4.12.A.4** For a linear transformation, L, from R^2 to R^2, there is a unique 2×2 matrix, A, such that $L(\vec{v}) = A\vec{v}$ for vectors in R^2. Conversely, for a given 2×2 matrix, A, the function $L(\vec{v}) = A\vec{v}$ is a linear transformation from R^2 to R^2.
- **4.12.A.5** Multiplication of a 2×2 transformation matrix, A, and a $2 \times n$ matrix of n input vectors gives a $2 \times n$ matrix of the n output vectors for the linear transformation $L(\vec{v}) = A\vec{v}$.

TOPIC 4.13

Matrices as Functions

Skills Focus: 1.B, 2.A, 3.A

LEARNING OBJECTIVES	ESSENTIAL KNOWLEDGE
4.13.A Determine the association between a linear transformation and a matrix.	▪ **4.13.A.1** The linear transformation mapping $\langle x, y\rangle$ to $\langle a_{11}x+a_{12}y, a_{21}x+a_{22}y\rangle$ is associated with the matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. ▪ **4.13.A.2** The mapping of the unit vectors in a linear transformation provides valuable information for determining the associated matrix. ▪ **4.13.A.3** The matrix associated with a linear transformation of vectors that maps every vector to the vector that is an angle θ counterclockwise rotation about the origin from the original vector is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. ▪ **4.13.A.4** The absolute value of the determinant of a 2×2 transformation matrix gives the magnitude of the dilation of regions in R^2 under the transformation.
4.13.B Determine the composition of two linear transformations.	▪ **4.13.B.1** The composition of two linear transformations is a linear transformation. ▪ **4.13.B.2** The matrix associated with the composition of two linear transformations is the product of the matrices associated with each linear transformation.
4.13.C Determine the inverse of a linear transformation.	▪ **4.13.C.1** Two linear transformations are inverses if their composition maps any vector to itself. ▪ **4.13.C.2** If a linear transformation, L, is given by $L(\vec{v})=A\vec{v}$, then its inverse transformation is given by $L^{-1}(\vec{v})=A^{-1}\vec{v}$, where A^{-1} is the inverse of the matrix A.

TOPIC 4.14

Matrices Modeling Contexts

Skills Focus: 1.C, 3.B, 3.C

LEARNING OBJECTIVES	ESSENTIAL KNOWLEDGE
4.14.A Construct a model of a scenario involving transitions between two states using matrices.	▪ **4.14.A.1** A contextual scenario can indicate the rate of transitions between states as percent changes. A matrix can be constructed based on these rates to model how states change over discrete intervals.
4.14.B Apply matrix models to predict future and past states for n transition steps.	▪ **4.14.B.1** The product of a matrix that models transitions between states and a corresponding state vector can predict future states. ▪ **4.14.B.2** Repeated multiplication of a matrix that models the transitions between states and corresponding resultant state vectors can predict the steady state, a distribution between states that does not change from one step to the next. ▪ **4.14.B.3** The product of the inverse of a matrix that models transitions between states and a corresponding state vector can predict past states.